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Quantum refrigerator driven by current noise

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Abstract – We proposed a scheme for the implementation of a "self-contained" quantum refrigerator system by three rf-SQUID qubits, or rather, flux-biased phase qubits. The three qubits play the role of the target, the refrigerator and the heat engine, respectively. We provide the three qubits with different effective temperatures, by imposing external current noises of different strengths. When the effective temperatures satisfy proper conditions, the target qubit could be cooled down. We also show that the efficiency of this system approaches the Carnot upper bound.

Introduction. – It is an interesting problem to discuss how small we can create a cooling machine and what would happen when quantum effects are taken into consideration, *e.g.*, whether a quantum refrigerator could exceed the classical Carnot efficiency. In practice, it is also a great challenge to obtain lower temperatures for better performance of quantum devices.

A lot of work has been done, both theoretically and experimentally [1–6], for the cooling of quantum systems. Most proposals require external fields for excitation or periodic control. The main mechanism is to lower down the population of the excited states, so as to get down the effective temperature T^{eff} of the small system, which is defined by treating the steady distribution of the system as a Gibbs one even when the whole system is not in equilibrium [2,7].

However, recently, Linden *et al.* proposed a "self-contained" refrigerator system, which has a heat engine inside to drive the whole system [8–10]. Here, "self-contained" means that it does not require any external source of work but only access to heat baths [10]. The system contains three qubits, which play the role of the target to be cooled (qubit 1), the refrigerator (qubit 2) and the heat engine (qubit 3), respectively. The three qubits contact independent reservoirs with different temperatures, and they interact through

$$\hat{V}=\tilde{g}(|010\rangle\langle 101|+|101\rangle\langle 010|)\equiv\hat{v}_C+\hat{v}_H.$$

$E_1+E_3=E_2$ is also required to guarantee the energy conservation, where E_α is the energy of each two-level system, $H_\alpha=E_\alpha|1\rangle_\alpha\langle 1|$.

The effect of $\hat{v}_C$ is to cool down qubit 1 (the target) by qubit 2 (the refrigerator), because we can see that $\hat{v}_C$ flips qubit 1 to the ground state and therefore lower down its energy. Reversely, $\hat{v}_H$ gives the heating effect. In order to make sure that $\hat{v}_C$ dominates, qubit 3 (the heat engine) is set to be hot enough. That is to say, the population of $|1\rangle$ of qubit 3 must be large enough, so as to enhance the cooling process. A clearer interpretation about this cooling mechanism would be shown below, *i.e.*, when $|\langle\hat{v}_C\rangle|>|\langle\hat{v}_H\rangle|$, the refrigerator works.

The main goal of our paper is to give a feasible scheme that could make this refrigerator realizable in the lab. Our model is composed of three rf-SQUID qubits, or rather, flux-biased phase qubits [11–13]. Josephson circuits techniques are relatively sophisticated nowadays, and we will see below that they have some unique merits to build this refrigerator system.

One of the obstacles to build this refrigerator is that, in experiments, it is hard to build systems with 3-body interaction directly. However, indirect 3-body interaction may arise from the transmission of basic 2-body interactions, if we impose proper resonant condition [14].

Another problem seems more difficult in the lab, *i.e.*, how to maintain the three microscopic qubits in *different* temperatures. In experiments, qubits are usually settled together and separate from each other at a distance of only several micrometers. It is hard to imagine that they have

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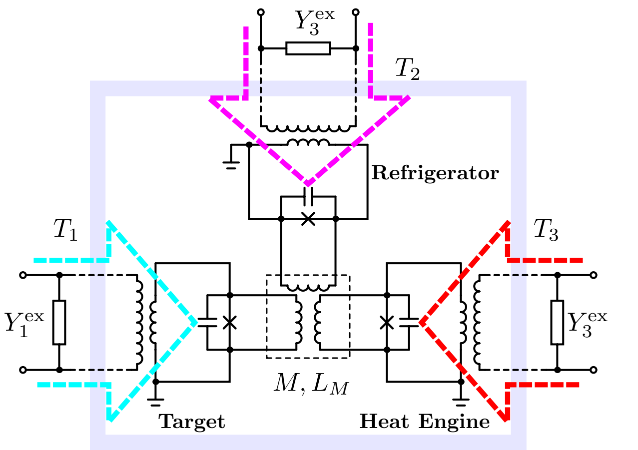

Fig. 1: (Color online) The design of the circuit. The part inside the dashed line square represents three identical mutual inductance coils. The big grey square denotes the region inside the mixing chamber of the dilution refrigerator. Y_α^{ex} is the effective admittance of external circuits.

different temperatures. We would solve this problem by utilizing the effective temperatures of the Nyquist current noises brought in from external circuits.

In experiments, Josephson circuits are usually thermally anchored to the mixing chamber in a dilution refrigerator at a temperature $T_{mix} \approx 10\,\mathrm{mK}$. We also have some control circuits outside the chamber at room temperature (see fig. 1). The effective admittances Y_α^{ex} of external circuits, which produce Ohmic heat, are the sources of current noises. These current noises are unavoidably brought to the qubits inside, and give rise to an effective Nyquist temperature different from T_{mix} (maybe as high as $\sim 300\,\mathrm{mK}$ [15]). We can control the strength of the current noise of each qubit by current filters so as to provide different temperatures as we need.

It is notable that recently Levy and Kosloff also proposed a good idea to overcome the 3-body interaction by replacing the cooling qubit with a classical noise source [16], and that also makes the refrigerator easier for implementation.

Proposal. – In this section, we show the configuration of our proposal. After a preliminary reduction, the Hamiltonian of the circuit can be reduced to that of three qubits with bipartite interactions.

The circuit design is shown in fig. 1. Flux bias for each qubit is provided by external noisy current, and we assume this is the main contribution to the dissipation in our system. The three qubits interact through mutual inductive coils overlaid together [12,13], as represented in the dashed line square in fig. 1. For simplicity, we assume the three mutual inductive coils are identical.

To get the quantum description of the system, we can write down the classical equations of motion of the circuits, then get the Lagrangian of the system, and finally, obtain the conjugate canonical momentum $\tilde{\Phi}_0 \cdot \hat{Q}_\alpha$ and the

$$U(\varphi) = E_J \left[1 - \cos\varphi - k(\varphi - \tilde{\varphi}_{\mathrm{bias}})^2\right]$$

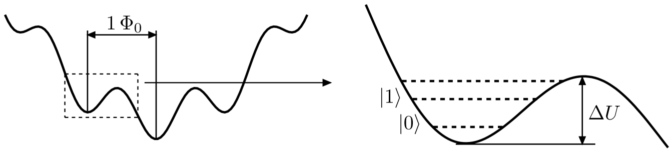

Fig. 2: Illustration of the potential of the Josephson circuit.

Hamiltonian of the circuits,

$$\hat{H}_{\mathrm{cir}} = \sum_{\alpha=1}^{3}\left[\frac{\hat{Q}_\alpha^2}{2C_\alpha} - E_J^\alpha \cos\hat{\varphi}_\alpha + \frac{\tilde{\Phi}_0^2}{2L_\alpha}(\hat{\varphi}_\alpha - \Phi_\alpha^{\mathrm{ext}}/\tilde{\Phi}_0)^2\right] + \frac{\tilde{\Phi}_0^2(L_M+M)}{(L_M+2M)(L_M-M)}\left[\sum_{\alpha=1}^{3}\frac{1}{2}\hat{\varphi}_\alpha^2 - \sum_{\alpha<\beta}\hat{\varphi}_\alpha\hat{\varphi}_\beta\right], \tag{1}$$

where $\tilde{\Phi}_0 = \Phi_0/2\pi$ and Φ_0 is the flux quantum. $\Phi_\alpha^{\mathrm{ext}}$ is the external flux imposed to the rf-SQUID loop. Q_α is the charge carried by the capacitance of the Josephson junction, and φ_α is the superconducting phase difference across each junction. L_α is the self-inductance of each rf-SQUID loop. L_M and M are the self and mutual inductances of the inductive coils, as denoted in fig. 1.

We set the parameters in such a way that each qubit works in a meta-stable cubic well approximately (see fig. 2). Move the origin of φ_α to the stable point of potential $U_\alpha(\varphi_\alpha)$, and expand it around the stable point to the third order, we can rewrite eq. (1) as

$$\begin{aligned}\hat{H} &= \sum_{\alpha=1}^{3}\hat{H}_\alpha + \hat{V}_{\mathrm{int}} \\ &= \sum_{\alpha=1}^{3}\left(\frac{\hat{p}_\alpha^2}{2m_\alpha} + \frac{m_\alpha\omega_\alpha^2\hat{x}_\alpha^2}{2} - \lambda\hat{x}_\alpha^3\right) + g\sum_{\alpha<\beta}\hat{x}_\alpha\hat{x}_\beta.\end{aligned} \tag{2}$$

m_α, ω_α, λ and the coupling constant g are determined from the parameters in eq. (1), like E_J^α, L, M, etc. $x_\alpha = \varphi_\alpha - \varphi_\alpha^{\mathrm{sta}}$ is the translated coordinate whose stable point is settled at the origin point.

Therefore, each circuit loop can be treated as a harmonic oscillator plus a cubic term as the perturbation,

$$\hat{H}_\alpha = \frac{\hat{p}_\alpha^2}{2m_\alpha} + \frac{m_\alpha\omega_\alpha^2\hat{x}_\alpha^2}{2} - \lambda\hat{x}_\alpha^3.$$

So we have $\hat{x}_\alpha = [\hbar/2m_\alpha\omega_\alpha]^{\frac{1}{2}}(\tilde{a}_\alpha + \tilde{a}_\alpha^\dagger)$, and $\tilde{a}_\alpha$ is the annihilation operator of the harmonic oscillator.

Corrections from the cubic term should be made to the energy levels and relevant eigen-states. And if we focus on the dynamics of the lowest two levels of the oscillator $\hat{H}_\alpha$, denoted by $|0\rangle_\alpha$ and $|1\rangle_\alpha$ hereafter, we obtain the low-energy effective Hamiltonian for each two-level qubit as $\hat{H}_\alpha^{\mathrm{TL}} = E_\alpha|1\rangle_\alpha\langle 1|$. Here, $E_\alpha \simeq 0.95\hbar\omega_\alpha$ is the modified energy of the qubit [12].

We can write down the Hamiltonian for the three interacting two-level systems,

$$\begin{aligned}\hat{H} &= \sum_{\alpha=1}^{3} E_\alpha |1\rangle_\alpha\langle 1| + g\sum_{\alpha<\beta}\hat{x}_\alpha\hat{x}_\beta \\ &= \sum \hat{H}_\alpha^{\mathrm{TL}} + \hat{V}_{\mathrm{int}},\end{aligned} \tag{3}$$

Effective Hamiltonian. – In this section, we start from eq. (3) and obtain an effective Hamiltonian with 3-body interaction, which is essential for our refrigerator as mentioned in the introduction.

We can obtain effective indirect interactions by treating $\hat{V}_{\mathrm{int}}$ as perturbation and imposing proper resonant condition. In order to obtain the effective Hamiltoian to describe indirect interactions of higher order, we turn to the interaction picture,

$$\begin{aligned}\mathcal{U}_I(t) &= \mathbf{T}\exp\left[-\frac{i}{\hbar}\int_0^t \mathrm{d}\tau H_I(\tau)\right] = \mathbf{1} - \frac{i}{\hbar}\int_0^t \mathrm{d}\tau H_I(\tau) \\ &\quad - \frac{1}{\hbar^2}\int_0^t \mathrm{d}\tau_1\int_0^{\tau_1}\mathrm{d}\tau_2 H_I(\tau_1)H_I(\tau_2) + \cdots \\ &= \mathbf{1} - \frac{i}{\hbar}H_{\mathrm{eff}}t + o(t^2).\end{aligned}$$

Let $t\to\infty$, the effective Hamiltonian is obtained as the linear term of the evolution operator. We can calculate the matrix elements $\langle\mathbf{n}|H_{\mathrm{eff}}|\mathbf{m}\rangle$ to obtain

$$\begin{aligned}H_{\mathrm{eff}} &= \sum_{\mathbf{n},\mathbf{m}}\langle\mathbf{n}|H_{\mathrm{eff}}|\mathbf{m}\rangle\,|\mathbf{n}\rangle\langle\mathbf{m}| \\ &= H_{\mathrm{eff}}^{(1)} + H_{\mathrm{eff}}^{(2)} + \cdots .\end{aligned} \tag{4}$$

Here $|\mathbf{n}\rangle = |n_1n_2n_3\rangle$ is the state of the three two-level qubits and $n_\alpha = 0,\,1$.

In the interaction picture,

$$\langle\mathbf{n}|H_I(t)|\mathbf{m}\rangle = \langle\mathbf{n}|\hat{V}_{\mathrm{int}}|\mathbf{m}\rangle\exp\left[\frac{i(E_\mathbf{n}-E_\mathbf{m})t}{\hbar}\right], \tag{5}$$

where $E_\mathbf{n}$ is the eigen-energy of $|\mathbf{n}\rangle$ and we denote $\Delta\omega_{\mathbf{nm}} = (E_\mathbf{n}-E_\mathbf{m})/\hbar$ for simplicity. Put the expression of $\langle\mathbf{n}|H_I(t)|\mathbf{m}\rangle$ into the integrals in $\mathcal{U}_I(t)$ and let the upper limit of the integral $t\to\infty$. We would see that not too many terms can survive, because the integrals of $e^{i\Delta\omega t}$ give out terms that contain

$$\begin{aligned}\lim_{t\to\infty}\left[e^{i\Delta\omega t}-1\right] &= \lim_{t\to\infty}\left[-2\sin^2\frac{\Delta\omega t}{2} + i\sin\Delta\omega t\right] \\ &= (-\Delta\omega^2 t\pi + i2\pi\Delta\omega)\delta(\Delta\omega).\end{aligned}$$

This is just the requirement of enery conservation. The only terms that survive must satisfy $E_\mathbf{n} = E_\mathbf{m}$. Remember that we impose a resonant condition $E_1 + E_3 = E_2$. Therefore, the only nonzero terms are two transition ones $\langle 101|H_{\mathrm{eff}}|010\rangle = \langle 010|H_{\mathrm{eff}}|101\rangle^*$ and eight diagonal ones $\langle\mathbf{n}|H_{\mathrm{eff}}|\mathbf{n}\rangle$.

We can calculate these transition amplitude up to the second order in the time-order expansion,

$$\begin{aligned}\langle\mathbf{n}|H_{\mathrm{eff}}^{(1)}|\mathbf{m}\rangle &= \langle\mathbf{n}|\hat{V}_{\mathrm{int}}|\mathbf{m}\rangle \equiv \hat{V}_{\mathrm{int}}^{\mathbf{nm}}, \\ \langle\mathbf{n}|H_{\mathrm{eff}}^{(2)}|\mathbf{m}\rangle &= \sum_{\mathbf{k}}^{E_\mathbf{k}\neq E}\frac{\hat{V}_{\mathrm{int}}^{\mathbf{nk}}\hat{V}_{\mathrm{int}}^{\mathbf{km}}}{E_\mathbf{k}-E}.\end{aligned} \tag{6}$$

Here $E_\mathbf{n} = E_\mathbf{m} = E$.

To carry out the calculation of $\hat{V}_{\mathrm{int}}^{\mathbf{nm}}$, we would come across terms like $\langle n_\alpha|\hat{x}_\alpha|n_\alpha\rangle$, which are not zero any more because we have taken anharmonic corrections into $|n_\alpha\rangle$. Perturbation up to the second order shows that

$$\hat{x}_\alpha^{01}\Big/\left[\frac{\hbar}{2m_\alpha\omega_\alpha}\right]^{\frac{1}{2}} \equiv c_\alpha^{01} = \langle 0|\tilde{a}_\alpha + \tilde{a}_\alpha^\dagger|1\rangle_\alpha \simeq 1 + \frac{0.096}{N},$$

$$c_\alpha^{00} \simeq \frac{1}{2\sqrt{3N_\alpha}}, \qquad c_\alpha^{11} \simeq \frac{3}{2\sqrt{3N_\alpha}},$$

where $N_\alpha = \Delta U_\alpha/\hbar\omega_\alpha$ and ΔU_α is the height of the cubic potential shown in fig. 2.

After all these preparations, we can get down to do the calculations. For the first order in $\mathcal{U}_I(t)$, since $\hat{V}_{\mathrm{int}} = g\sum\hat{x}_\alpha\hat{x}_\beta$ involves only two-body interaction, clearly we have $\langle 101|\hat{V}_{\mathrm{int}}|010\rangle = 0$ while the diagonal terms survive,

$$D_\mathbf{n} \equiv \langle\mathbf{n}|H_{\mathrm{eff}}^{(1)}|\mathbf{n}\rangle = g\sum_{\alpha<\beta}\hat{x}_\alpha^{n_\alpha n_\alpha}\hat{x}_\beta^{n_\beta n_\beta}. \tag{7}$$

For the second order, we only calculate the transition terms. We must calculate terms like $\langle 101|\hat{x}_\alpha\hat{x}_\beta|\mathbf{k}\rangle\times\langle\mathbf{k}|\hat{x}_{\alpha'}\hat{x}_{\beta'}|010\rangle$ for all $|\mathbf{k}\rangle = |k_1k_2k_3\rangle$ that satisfy $E_\mathbf{k}\neq E_{101}$. And then we get

$$\begin{aligned}\tilde{g} &\equiv \langle 101|H_{\mathrm{eff}}^{(2)}|010\rangle \\ &= g^2\hat{x}_1^{10}\hat{x}_2^{10}\hat{x}_3^{10}\times\left[\frac{(\hat{x}_2^{11}-\hat{x}_2^{00})-(\hat{x}_3^{11}-\hat{x}_3^{00})}{E_1}\right. \\ &\quad\left. + \frac{(\hat{x}_1^{11}-\hat{x}_1^{00})+(\hat{x}_3^{11}-\hat{x}_3^{00})}{E_2} + \frac{(\hat{x}_2^{11}-\hat{x}_2^{00})-(\hat{x}_1^{11}-\hat{x}_1^{00})}{E_3}\right].\end{aligned}$$

Put $D_\mathbf{n}$ and $\tilde{g}$ back into eq. (4), we arrive at the effective Hamiltonian of the three qubits as

$$H_{\mathrm{eff}} = \sum_\mathbf{n} D_\mathbf{n}|\mathbf{n}\rangle\langle\mathbf{n}| + \tilde{g}(|010\rangle\langle 101| + |101\rangle\langle 010|).$$

We have eight $D_\mathbf{n}$'s here. For simplicity of the discussion in the paper, we reorganize the diagonal terms as the summation of 1-, 2-, 3-body operators,

$$\begin{aligned}H_{\mathrm{eff}} &= \sum_\alpha D^{(\alpha)}\hat{n}_\alpha + \sum_{\alpha<\beta}D^{(\alpha\beta)}\hat{n}_\alpha\hat{n}_\beta + D^{(123)}\hat{n}_1\hat{n}_2\hat{n}_3 \\ &\quad + \tilde{g}(|010\rangle\langle 101| + |101\rangle\langle 010|).\end{aligned} \tag{8}$$

The relation between $D_{\mathbf{n}}$ and $D^{(\cdots)}$ is,

$$\begin{aligned}
&D_{100}-D_{000}=D^{(1)}, \qquad D_{010}-D_{000}=D^{(2)},\\
&D_{001}-D_{000}=D^{(3)},\\
&D_{110}-D_{000}=D^{(1)}+D^{(2)}+D^{(12)},\\
&D_{011}-D_{000}=D^{(2)}+D^{(3)}+D^{(23)},\\
&D_{101}-D_{000}=D^{(1)}+D^{(3)}+D^{(13)},\\
&D_{111}-D_{000}=\sum D^{(\alpha)}+\sum D^{(\alpha\beta)}+D^{(123)}.
\end{aligned}$$

Remember that the coupling constant g in eq. (3) comes from the strength of inductive coupling, and it can be tuned in experiments. We can control the properties of the refrigerator by tuning relevant parameters.

Now we obtain the Hamiltonian of three two-level qubits with their effective interaction in the Schrödinger picture, from eq. (1),

$$\mathcal{H}_{\mathrm{S}}=\sum \hat{H}_{\alpha}^{\mathrm{TL}}+H_{\mathrm{eff}}. \tag{9}$$

Dissipation. – We have got the effective Hamiltonian of the three qubits with 3-body interaction, which is essential for our refrigerator, as mentioned in the introduction. In experiments, the current bias is not ideal but with noises. Equivalently, current noises can be treated as heat sources and that provides energy for the thermal mechine. In this section, we treat the noisy qubits by master equation and give the steady solutions.

Master equation. Dissipation due to external current noise has been well discussed in the literature [17–22]. In our system, the flux bias is provided through $\Phi_{\alpha}^{\mathrm{ext}}=L_{\alpha}^{\mathrm{ext}}\tilde{I}_{\alpha}=L_{\alpha}^{\mathrm{ext}}[I_{\alpha}+\hat{i}_{\alpha}(t)]$. $L_{\alpha}^{\mathrm{ext}}$ comes from an external coil that provides flux bias for each qubit, and $\hat{i}_{\alpha}(t)$ is the noise that satisfies $\langle\hat{i}_{\alpha}(t)\rangle=0$. Tracing back to the single qubit term in the Hamiltonian of the Josephson circuit eq. (1), we can figure out that the qubits interact with the current noise through an interaction term $\mathcal{H}_{\mathrm{int}}=\sum_{\alpha}\gamma_{\alpha}\hat{x}_{\alpha}\cdot\hat{i}_{\alpha}$, where γ_{α} collects the parameters.

Hereafter, we take $\hat{x}_{\alpha}\simeq[\hbar/2m_{\alpha}\omega_{\alpha}]^{1/2}(a_{\alpha}+a_{\alpha}^{\dagger})$, where $a_{\alpha}^{\dagger}\equiv|1\rangle_{\alpha}\langle 0|$ and $a_{\alpha}\equiv|1\rangle_{\alpha}\langle 0|$ are the raising and lowering operator of each qubit.

The total Hamiltonian of the three qubits, reservoirs and their interactions is

$$\mathcal{H}_{\mathrm{SB}}=\mathcal{H}_{\mathrm{S}}+\mathcal{H}_{\mathrm{B}}+\mathcal{H}_{\mathrm{int}}, \tag{10}$$

$\mathcal{H}_{\mathrm{B}}$ is usually modeled as a collection of harmonic oscillators phenomenologically, and here we have three independent harmonic baths.

Following the method in refs. [21,22], we write down the master equation of the system, after a complicated but straightforward reduction that mainly includes Born-Markov approximation and RWA. The master equation of the three-qubit system in the interaction picture is

$$\begin{aligned}
\partial_t\rho \;=\; &-\frac{i}{\hbar}[H_{\mathrm{eff}},\rho]+\sum_{\alpha}\mathcal{D}_{\alpha}\rho=-\frac{i}{\hbar}[H_{\mathrm{eff}},\rho]\\
&+\sum_{\alpha}\Gamma_{\alpha}^{-}\left[2a_{\alpha}\rho a_{\alpha}^{\dagger}-\{a_{\alpha}^{\dagger}a_{\alpha},\rho\}_{+}\right]\\
&+\sum_{\alpha}\Gamma_{\alpha}^{+}\left[2a_{\alpha}^{\dagger}\rho a_{\alpha}-\{a_{\alpha}a_{\alpha}^{\dagger},\rho\}_{+}\right].
\end{aligned} \tag{11}$$

Here, $a_{\alpha}^{\dagger}=|1\rangle_{\alpha}\langle 0|$ and $a_{\alpha}=|0\rangle_{\alpha}\langle 1|$ are the corresponding raising and lowering operators of each qubit, as we mentioned before. $\Gamma_{\alpha}^{\pm}$ represents the absorbing and emitting rate of each two-level system, respectively,

$$\begin{aligned}
\Gamma_{\alpha}^{\pm} \;&=\; \frac{\gamma_{\alpha}^{2}}{2m_{\alpha}\hbar\omega_{\alpha}}\int_{-\infty}^{\infty}\mathrm{d}\tau\, e^{\mp iE_{\alpha}\tau/\hbar}\langle\hat{i}(\tau)\hat{i}(0)\rangle\\
&=\; \frac{\gamma_{\alpha}^{2}}{2m_{\alpha}\hbar\omega_{\alpha}}S_{I}^{\alpha}(\mp E_{\alpha}/\hbar).
\end{aligned} \tag{12}$$

Here $S_{I}^{\alpha}(\omega)$ is the power spectral of current noise reflecting the dissipative impedance of the circuit [17,21],

$$S_{I}^{\alpha}(\omega)=2\hbar\omega\,\mathrm{Re}\,Y_{\alpha}^{\mathrm{ex}}(\omega)\,[N_{\alpha}(\omega)+1], \tag{13}$$

where $Y_{\alpha}^{\mathrm{ex}}(\omega)$ is the effective admittance of the external circuit, and it can be measured in experiment. $Y_{\alpha}^{\mathrm{ex}}(\omega)$ produces Ohmic heat, which is actually the macroscopic effect of Nyquist noise. And this is also the reason why we must consider the noise temperature which is different from T_{mix} in the chamber. $N_{\alpha}(\omega)=[\exp(\hbar\omega/kT_{\alpha})-1]^{-1}$ is the Planck distribution. When $\hbar\omega\ll kT_{\alpha}$, we have $S_{I}^{\alpha}(\omega)\simeq 2kT_{\alpha}Y_{\alpha}(\omega)$, which is the Nyquist formula.

Thus, we can see the absorbing and emitting rate $\Gamma_{\alpha}^{\pm}$ relate to the noise temperatures,

$$\Gamma_{\alpha}^{-}=\Gamma_{\alpha}(N_{\alpha}+1), \qquad \Gamma_{\alpha}^{+}=\Gamma_{\alpha}N_{\alpha}.$$

We must emphasize that the T_{α} in N_{α} is the Nyquist temperature of the current noise coming from external circuits, but not T_{mix}, as mentioned in introduction. The strength of the current noise can be controlled by filters, and that also determines the noise temperature T_{α}. Thus, we can provide the three qubits with independent reservoirs of different temperatures. We will see below that proper control of the noise temperature T_{α} makes our refrigerator work.

Steady solution. We concentrate on the equilibrium behaviour of the system here, and especially, we want to obtain the final steady distribution $\langle\hat{n}_{\alpha}\rangle$ of each single qubit, and then to get the effective temperatures.

We can obtain a closed set of eight independent linear equations about $\langle\hat{n}_{\alpha}\rangle$, $\langle\hat{n}_{\alpha}\hat{n}_{\beta}\rangle$, $\langle\hat{n}_1\hat{n}_2\hat{n}_3\rangle$ and $\langle\Delta v\rangle\equiv\tilde{g}\langle a_1a_2^{\dagger}a_3\rangle-\mathrm{h.c.}=\langle\hat{v}_C\rangle-\langle\hat{v}_H\rangle$. By solving the equations, the steady solution is derived as follows:

$$\langle\hat{n}_{\alpha}\rangle=\frac{N_{\alpha}}{2N_{\alpha}+1}+\frac{(-1)^{\alpha}\,\langle\Delta v\rangle/i}{2\Gamma_{\alpha}(2N_{\alpha}+1)}. \tag{14}$$

Here, $\langle \Delta v \rangle$ comes from the interaction H_{eff},

$$\begin{aligned}\langle \Delta v \rangle &= \frac{i\tilde{g}^2 G}{X_1 + \tilde{g}^2(X_2 + X_3)}(N_1N_3 - N_2 - N_1N_2 - N_2N_3) \\ &\equiv i\xi(N_1N_3 - N_2 - N_1N_2 - N_2N_3), \qquad (15)\end{aligned}$$

and

$$\begin{aligned}G &= 4\Gamma_1\Gamma_2\Gamma_3\Theta_s\Theta_p, \\ X_1 &= 2\Lambda_1\Lambda_2\Lambda_3\Theta_p(\Theta_D^2 + \Theta_s^2), \\ X_2 &= \Theta_p\left[4\Lambda_1\Lambda_2\Lambda_3 + \sum_{\alpha<\beta}\Lambda_\alpha\Lambda_\beta(\Lambda_\alpha + \Lambda_\beta)\right], \\ X_3 &= -\Gamma_1\Gamma_2\Lambda_1\Lambda_2(\Theta_s^2 - \Lambda_3^2) + \Gamma_1\Gamma_3\Lambda_1\Lambda_3(\Theta_s^2 - \Lambda_2^2) \\ &\quad - \Gamma_2\Gamma_3\Lambda_2\Lambda_3(\Theta_s^2 - \Lambda_1^2),\end{aligned}$$

where $\Theta_p \equiv \prod_{\alpha<\beta}(\Lambda_\alpha + \Lambda_\beta)$.

Here we define $\Theta_s \equiv \sum_\alpha \Lambda_\alpha$ and $\Lambda_\alpha \equiv \Gamma_\alpha(2N_\alpha + 1)$. Indeed Λ_α is the decay rate. The contribution of diagonal terms is reflected in

$$\begin{aligned}\Theta_D &\equiv D^{(1)} - D^{(2)} + D^{(3)} + D^{(13)} \\ &= \langle 101|H_{\text{eff}}|101\rangle - \langle 010|H_{\text{eff}}|010\rangle. \qquad (16)\end{aligned}$$

The steady solution of $\langle \hat{n}_\alpha \rangle$ contains two terms, and it is the population probability of $|1\rangle_\alpha$. When there is no interaction between the three qubits, *i.e.*, $\tilde{g} = 0$, the second term disappears. That means they decay into Boltzmann distributions, respectively, due to the weak coupling with each reservoir, as described by the first term in eq. (14). $P_{|0\rangle_\alpha} : P_{|1\rangle_\alpha} = 1 : e^{-\beta E_\alpha}$. This is consistent with the results of a single qubit in a Markovian bath [22].

With the presence of the 3-body interaction H_{eff}, the population $\langle \hat{n}_\alpha \rangle$ is changed, as described by the second term in eq. (14). Correspondingly, the effective temperatures would be changed as long as $\langle \Delta v \rangle \neq 0$.

Discussion. – We have got the steady solution of $\langle \hat{n}_\alpha \rangle$ in eq. (14). In this section, we talk about the physical meaning of the steady solution further to give the cooling condition of the refrigerator. We also get the efficiency of this thermal machine. At last, we talk about the measurement and some constrictions in practical experiments.

Cooling condition. We are going to discuss the physical meaning of the solution and give the cooling condition of the refrigerator now.

When we say that we want to make the system run as a refrigerator in order to cool down qubit 1, we have actually implied that the initial temperature T_1 is the lowest, otherwise, it could be cooled through heat transportion. Moreover, we must make sure $\langle \Delta v \rangle / i > 0$ so as to lower down $\langle \hat{n}_1 \rangle$. That means, the system is staying at lower temperature and less likely to be excited.

We can check that each term in ξ in eq. (15) is positive. Thus, we require that $N_1N_3 - N_2 - N_1N_2 - N_2N_3 > 0$, and that gives the cooling condition,

$$\begin{aligned}&(\beta_1 - \beta_2)E_1 < (\beta_2 - \beta_3)E_3, \\ \text{or,}\quad &\frac{E_1}{E_3} < \frac{\beta_2 - \beta_3}{\beta_1 - \beta_2}. \qquad (17)\end{aligned}$$

Remember that $E_1 + E_3 = E_2$. As we know T_1 should be the lowest, we must have $T_1 < T_2 < T_3$.

Notice that $\langle \Delta v \rangle / i > 0$ also means $|\langle \hat{v}_C \rangle| > |\langle \hat{v}_H \rangle|$. That gives a well physical meaning that the cooling process dominates over heating, as we mentioned in the beginning about the cooling mechanism of this system.

The transition terms give rise to the cooling effect, while the diagonal terms weaken the cooling effect, but would not eliminate it, as seen from the expression of X_1 and Θ_D in eq. (15). Qubit 1 can be cooled if and only if eq. (17) is satisfied.

Reliability. We have shown that these three qubits could work as a refrigerator under the resonant condition $E_1 + E_3 = E_2$. Here we analyse the reliability when this condition is slightly broken.

Assume we have $E_1 + E_3 = E_2 + \delta\varepsilon$, we can rewrite the Hamiltonian eq. (3) as

$$\hat{H}' = \sum_\alpha \hbar\omega_\alpha a_\alpha^\dagger a_\alpha + \left(\hat{V}_{\text{int}} - \delta\varepsilon a_2^\dagger a_2\right), \qquad (18)$$

where $\hbar\omega_{1,3} = E_{1,3}$, while $\hbar\omega_2 = E_2 + \delta\varepsilon$. We can derive the effective Hamiltonian by the method we used above in the interaction picture of $H_0 = \sum_\alpha \hbar\omega_\alpha a_\alpha^\dagger a_\alpha$. After repeating the calculation as before, we find that the detuning term $\delta\varepsilon a_2^\dagger a_2 = \delta\varepsilon \hat{n}_2$ does not change our previous result too much,

$$\begin{aligned}H'_{\text{eff}} &= \sum_\alpha D^{(\alpha)}\hat{n}_\alpha + \sum_{\alpha<\beta} D^{(\alpha\beta)}\hat{n}_\alpha\hat{n}_\beta + D^{(123)}\hat{n}_1\hat{n}_2\hat{n}_3 \\ &\quad + \tilde{g}(|010\rangle\langle 101| + |101\rangle\langle 010|) - \delta\varepsilon\hat{n}_2. \qquad (19)\end{aligned}$$

When the detuning $\delta\varepsilon$ is too large comparing with the coupling energy, the interaction terms in H'_{eff} are suppressed and the qubits do not interact with each other any more.

Notice that the added term can be absorbed into $D^{(2)}$. Therefore, the steady solution has the same form with that of the unperturbed case, only with $D^{(2)} \to D^{(2)} - \delta\varepsilon$. Now the diagonal contribution becomes

$$\Theta'_D = \langle 101|H_{\text{eff}}|101\rangle - \langle 010|H_{\text{eff}}|010\rangle + \delta\varepsilon. \qquad (20)$$

When the detuning $\delta\varepsilon$ is quite large, $\Theta_D \to \infty$, that would suppress Δv in $\hat{n}_\alpha$ to zero. This is consistent with the argument that the interaction would turn off when the detuning is large.

On the other hand, it is amazing to notice that proper detuning could eliminate the weakening effect of the diagonal terms completely to zero, as long as we take $\delta\varepsilon = D_{010} - D_{101}$.

Efficiency. In order to compute the cooling efficiency of the refrigerator, we have to compare the amounts of heat exchange of the target (qubit 1) and the heat engine (qubit 3) with their environments. The unitary term in the master equation eq. (11) represents the contribution of doing works between the three qubits, and the second dissipative one represents the heat exchange with the environment. Therefore, we can get the heat exchange of each qubit with their environments per unit time by

$$\mathcal{Q}_\alpha = \mathrm{Tr}[E_\alpha \hat{n}_\alpha \cdot \mathcal{D}_\alpha \rho] = (-1)^{\alpha+1} E_\alpha \langle \Delta v \rangle / i.$$

Now we arrive at the interesting result that the efficiency of the system is

$$\eta^{\mathcal{Q}} = \frac{\mathcal{Q}_1}{\mathcal{Q}_3} = \frac{E_1}{E_3} < \frac{1-\frac{T_2}{T_3}}{\frac{T_2}{T_1}-1} \equiv \eta^{\mathcal{Q}}_{\max}. \qquad (21)$$

Remember that $T_1 < T_2 < T_3$. This is exactly the same with the results in previous work [9]. In their work, the authors of [9] also figure out that this is the upper bound on the efficiency of any such engine running between three reservoirs which extracts heat from the bath at T_1 using a supply of heat from the bath at T_3.

From the results and discussion above, we can see that, by controlling the effective Nyquist temperatures, we can provide the three qubits with different reservoirs and make the refrigerator work well.

Measurement. In experiments, what we should do is to input proper noisy current to each qubit, and measure whether each qubit is at $|1\rangle_\alpha$ after a relaxation time $< 1\,\mu s$. Multiple repeated trials give $\langle \hat{n}_\alpha \rangle$, and then we can get the effective temperature of each qubit.

Each single measurement can be done as follows. The height of the barrier ΔU can be tuned by the flux bias. When the barrier is tuned low enough, the oscillator would tunnel out to the next well if it was in state $|1\rangle_\alpha$ (see fig. 2), and that causes a flux change of $\sim 1\Phi_0$ in the qubit loop which can be detected by a SQUID. While this would not happen if the original state was $|0\rangle_\alpha$. This is usually done by imposing a current pulse $\sim 2\,$ns to each qubit [23], and it is much faster than the evolution of the interaction. Moreover, since ΔU is changed, the resonant condition is greatly broken, and the interaction stops. This method has been used for the tomography of entanglement [11].

We should talk about some restriction that may be required in experiments. First, the highest temperature T_3 must not be too high comparing with the excitation energy $E_3 \sim 10\,\mathrm{GHz} \sim 1\,\mathrm{K}$, otherwise, population of higher energy levels must be taken into account, and we cannot treat our system simply as a two-level system. Second, if we greatly suppress the external noise of qubit 1 in order to make T_1 lower ($< 50\,\mathrm{mK}$), the low-frequency $1/f$ noise would dominate the dissipation. Therefore, our analysis above based on Markovian approximation is invalid.

It is interesting that the whole system could obtain power from noises that we usually dislike. We emphasize that the current noises come from outside, thus they are stable and independent from the thermal environment in the mixing chamber. Besides, the noise sources of the qubits are independent from each other. Otherwise, the whole system would finally achieve thermal equilibrium and each part has the same temperature.

Summary. – In this paper, we proposed a Josephson circuit system to implement a quantum refrigerator. By controlling the different strengths of the noises pouring into the qubits, we show that the refrigerator could surely cool down the target. The efficiency of this heat machine is no greater than the Carnot up-bound. We believe our proposal is realizable with the present technology. More general case of noises, especially the $1/f$ type, should be discussed in future works.

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