Quantum correlations in topological quantum phase transitions

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We study the quantum correlations in a two-dimensional system that possesses a topological quantum phase transition. The quantumness of two-body correlations is measured by quantum discord. We calculate both the correlation of two local spins and that between an arbitrary spin and the rest of the lattice. It is notable that local spins are classically correlated, while the quantum correlation is hidden in the global lattice. This is different from other systems which are not topologically ordered. Moreover, the mutual information and global quantum discord show critical behavior in the topological quantum phase transition.

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I. INTRODUCTION

Topological phase is a new kind of order that cannot be described by symmetry-breaking theory [1]. A typical example is the quantum Hall system, which exhibits a lot of amazing properties, such as topological degeneracy and fractional statistical behaviors. Especially, the property of topological protection may lead to a new way of performing quantum computation [2].

Different from the quantum Hall system, the Kitaev toric code model is an exactly solvable spin lattice model that is topologically ordered [3]. The system is immune to small perturbations. The breaking down of the topological phase happens through a quantum phase transition [4]. A lot has been studied about the topological quantum phase transition, especially about the toric code model in the presence of a magnetic field [5–9].

Concepts of quantum information have been applied in the study of quantum phase transition, such as entanglement and fidelity [10,11]. Here, we are interested in the correlations in the topological phase, because the magic power of quantum computation is rooted in the strange nonclassical correlations.

Entanglement is the most important nonclassical correlation in quantum-information processing, such as quantum teleportation [12]. However, some separable states also have properties that are not achievable by classical methods [13]. Recent results suggest that these correlations may also take effect in quantum computation. The classification of nonclassical correlations and their effects still remain unclear [14].

Quantum discord is a measurement of the "quantumness" of a pairwise correlation [13,15]. It is based on the fact that the mutual information has two equivalent definitions in the classical world, while their quantum generations are not equivalent. Quantum discord is defined as the minimum of their difference and measures how "quantum" the correlation is. Besides entangled states, some separable states also have nonzero quantum discord, which means they are nonclassical. Recent studies suggest that quantum discord, but not entanglement, may be responsible for mixed-state computation [16,17].

Quantum discord in quantum phase transition has been studied in one-dimensional (1D) systems, such as the XXZ chain and some other Z_2-symmetric 1D spin models [18,19]. During the phase transition, quantum discord shows critical behavior at the phase transition point. In a study in the thermal Heisenberg system, quantum discord also demonstrated some different behavior from entanglement [20].

In this paper, we study the correlations in the Castelnovo-Chamon model [21], which is a two-dimensional (2D) system that shows a quantum phase transition from a topologically ordered phase to a magnetized one. It is a deformation of the toric code model and possesses higher symmetry than the 1D models mentioned above. Both local spin-spin correlation and that between a spin and the rest of the whole lattice are calculated. It is notable that in such a topologically ordered system, quantum discord of local spins is always zero in both phases, which means the local correlations are completely classical. However, the correlation between a local spin and the rest of the lattice behaves more like a pairwise entangled pure state, and the quantum discord signals the critical point. This is different from previous studies in other models [18–20]. Our results shows that in a topologically ordered system, the quantum correlation is hidden in the lattice globally by the high symmetry of the system.

Besides, we calculated the mutual information of the correlations in the system. It was pointed out that in topologically ordered systems, in which there is no local order parameter, the topological quantum phase transition can be signaled by local properties like the reduced fidelity of two spins, and it is even more sensitive than the global fidelity [22,23]. Here, we calculate the mutual information, both of the global correlation and that of two local spins. We see that the mutual information could also characterize the critical behavior.

The paper is organized as follows. In Sec. II, we briefly review the concept of quantum discord and its basic properties. In Sec. III, we introduce the Castelnovo-Chamon model. We give the ground state and explain how it can be mapped to the classical Ising model. In Sec. IV, we calculate the quantum discord of local spin-spin correlations and also that of a local spin with the rest of the whole lattice. Our conclusion is drawn in Sec. V.

II. QUANTUM DISCORD

Quantum discord can be used as a measure of the quantumness of a pairwise correlation [13]. In this part, we briefly introduce this concept.

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Information is the average amount of uncertainty that can be eliminated after we get a measurement result. Mutual information $\mathcal{I}(A:B)$ describes the information about A we gain after the measurement of B, or rather, the information that A and B have in common [24]. In classical world, there are two equivalent definitions of mutual information:

$$\mathcal{I}(A:B) = H(A) + H(B) - H(A,B),$$
$$\mathcal{J}(A:B) = H(A) - H(A|B),$$

with the Shannon entropy $H(\cdot) = -\sum_i p_i \log_2 p_i$. $H(A|B)$ is the conditional entropy, which is a measure of how uncertain we are about A, on average, when B is known. The Bayes law tells us that $p(a_i, b_j) = p(a_i|b_j)p(b_j) = p(b_j|a_i)p(a_i)$, where $p(a_i|b_j)$ is the conditional probability that describes the probability to get a_i when we know the value of B is b_j. That guarantees the equivalence of the two definitions above.

However, things are different when we generalize the concepts to the quantum world. We can get the quantum version of $\mathcal{I}(A:B)$ easily by replacing H with von Neumann entropy $S(\rho) = -\mathrm{tr}(\rho \log_2 \rho)$. While the concept of conditional entropy in fact implicitly calls for a measurement of B. To get the corresponding $\mathcal{J}(A:B)$, we have to choose a set of one-dimensional projectors $\{\hat{\Pi}_i^B\}$ to imply projective measurements on the system B. The state of the system after measurement is $\rho_i = \hat{\Pi}_i^B \rho_{AB} \hat{\Pi}_i^B / p_i$, where $p_i = \mathrm{tr}(\hat{\Pi}_i^B \rho_{AB} \hat{\Pi}_i^B)$. With the knowledge we gain after the measurement, we get

$$\begin{aligned}\mathcal{J}\big(A\big|\{\hat{\Pi}_i^B\}\big) &= S(\rho_A) - S\big(\rho_{AB}\big|\{\hat{\Pi}_i^B\}\big) \\ &= S(\rho_A) - \sum_i p_i S(\rho_i). \end{aligned} \tag{1}$$

The value of $\mathcal{J}(A|\{\hat{\Pi}_i^B\})$ depends on the choice of $\{\hat{\Pi}_i^B\}$, i.e., how we measure B, and therefore it may not be equal to $\mathcal{I}(A:B)$ any more. Quantum discord is defined as the minimum of their difference,

$$D(\rho) = \min\big[\mathcal{I}(A:B) - \mathcal{J}\big(A\big|\{\hat{\Pi}_i^B\}\big)\big]. \tag{2}$$

$\mathcal{J}(A|\{\hat{\Pi}_i^B\})$ describes the amount of information of A achievable by projective measurements on B. It can be proved that $D(\rho) \geqslant 0$. From the derivation above, we see that the quantum discord of classical correlations should be zero. We can use quantum discord as a measure of the "quantumness" of a two-body correlation.

The quantum discord of states that contain entanglement is obviously nonzero. However, it should be emphasized that not all separable states are classical under the definition of quantum discord. A simple example is $\rho = (|00\rangle\langle 00| + |++\rangle\langle ++|)/2$, where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. The quantum discord is nonzero. The information inside the state cannot be fully extracted just by local projective measurements.

However, if there exists a set of 1D projectors $\{\tilde{\Pi}_i^B\}$ such that ρ_{AB} can be written in the form of

$$\rho_{AB} = \sum p_i \rho_i^A \otimes \tilde{\Pi}_i^B, \tag{3}$$

then $\rho_B = \sum p_i \tilde{\Pi}_i^B$, and we can get

$$\begin{aligned}\mathcal{J}\big(\rho_{AB}\big|\{\tilde{\Pi}_B^i\}\big) &= S(A) - \sum p_i S\big(\rho_i^A\big) \\ &= S(A) - \sum p_i \lambda_k^i \log_2 \lambda_k^i \\ &= S(A) + S(B) - S(AB) = \mathcal{I}(\rho_{AB}),\end{aligned}$$

where λ_k^i is an eigenvalue of ρ_i^A. That is to say, when ρ_{AB} has the form of Eq. (3), the projective measurements $\{\tilde{\Pi}_i^B\}$ do not destroy the information of ρ_{AB}, therefore, the mutual information $\mathcal{J}$ which is gained after the measurements achieves the maximum $\mathcal{I}$, and the quantum discord is zero. We believe this is also a necessary condition.

III. CASTELNOVO-CHAMON MODEL

Castelnovo and Chamon proposed a model that shows topological quantum phase transition [21]. It is a deformation of the Kitaev toric code model. The Hamiltonian is

$$H = -\lambda_0 \sum_p B_p - \lambda_1 \sum_s A_s + \lambda_1 \sum_s \exp\left(-\beta \sum_{i\in s} \hat{\sigma}_i^z\right), \tag{4}$$

where $\lambda_{0,1} > 0$, $A_s = \prod_{i\in s} \hat{\sigma}_i^x$ and $B_p = \prod_{i\in p} \hat{\sigma}_i^z$ are the *star* and *plaquette operators* in the toric code model, respectively. β is a coupling constant. The star operator A_s acts on the four spins around the vertex s, while the plaquette operator B_p acts on the four spins on the edges of the plaquette q, as shown in Fig. 1. We consider the problem under the torus boundary condition.

The ground state can be written down analytically. We give the state in the topological sector that contains the fully magnetized state $|0\rangle = |\uparrow\uparrow\uparrow\cdots\uparrow\rangle$ as

$$|\mathrm{g.s.}(\beta)\rangle = Z(\beta)^{-\frac{1}{2}} \sum_{g\in G} \exp\left[\beta \sum_i \sigma_i^z(g)/2\right] g|0\rangle, \tag{5}$$

with $Z(\beta) = \sum_{g\in G} \exp[\beta \sum_i \sigma_i^z(g)]$. G is the Abelian group generated by the star operators $\{A_s\}$,

$$G = \left\{g|g = \prod_s A_s^{n_s}, n_s = 0, 1\right\}, \tag{6}$$

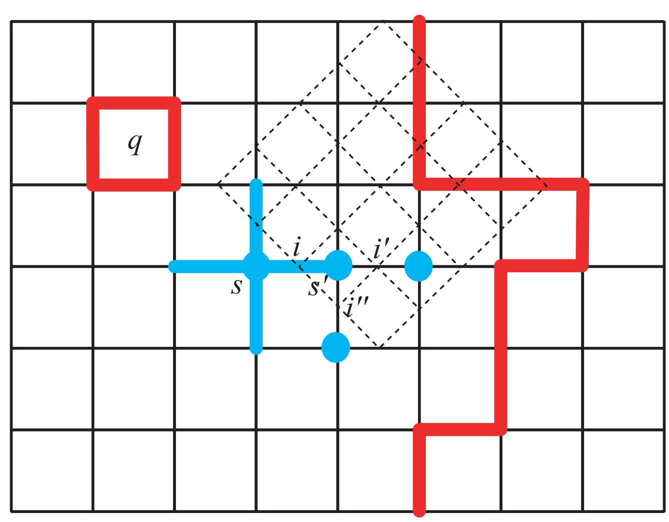

FIG. 1. (Color online). Demonstration of the model. Spins lie on the edges (like i, i', i''). The nearest spins i, i' become next-nearest in the dual lattice (the dashed line). The red plaquette and the blue cross represent the plaquette and star operators. The red heavy line across the lattices represents a product of σ_i^z along the nontrivial loop on a torus. The system is invariant under this transformation.

where the product contains terms generated from *all* the star operators. So $g|0\rangle$ contains separable spins taking the form like $|011110\cdots 0\rangle$, and we can denote each $g|0\rangle$ by a corresponding binary number as $|x\rangle$ (as what we do in the following). $\sigma_i^z(g)$ is the value of spin at site i in state $g|0\rangle$. The sum in the exponential term in fact counts the total magnetic polarization of $g|0\rangle$.

It may not be obvious to get Eq. (5) directly. We can just put it back into Eq. (4) and it can be easily checked. When $\beta = 0$, the model reduces to the toric code model. When $\beta \to \infty$, the ground state becomes the fully magnetized reference state $|0\rangle$. At $\beta_c = (1/2)\ln(\sqrt{2}+1)$, there is a second-order topological quantum phase transition, according to the study of topological entropy [21] and fidelity [22,23] in this model.

Furthermore, the value of $\sigma_i^z(g)$ (note that there is no hat, it is just an integer number relating to g) is actually determined by whether the two ends (s and s' in Fig. 1) of the ith edge are acted by $A_{s(s')}$ or not. As $A_s^2 = \mathbf{1}$, elements of G can be represented as a configuration of $\{\theta_s\}$, where $\theta_s = +1$ means A_s acts on vertex s, while $\theta_s = -1$ means not. So we get $\sigma_i^z(g) = \theta_s\theta_{s'}$. The normalizer in the ground state Eq. (5) is

$$Z(\beta) = \sum_{\{\theta_s\}} \exp\left[\beta \sum_{\langle ss'\rangle} \theta_s\theta_{s'}\right], \tag{7}$$

which is just the canonical partition function of the 2D classical Ising model without an external field, with the Hamiltonian $H_{\text{Ising}} = -\sum_{\langle ss'\rangle}\theta_s\theta_{s'}$.

As an example, we can calculate the correlation function as

$$\begin{aligned}\langle \text{g.s.}|\hat{\sigma}_i^z|\text{g.s.}\rangle &= \sum_{g\in G}\sigma_i^z(g)\exp\left[\beta\sum_j \sigma_j^z(g)\right]\\ &= \sum_{\{\theta_s\}}\theta_s\theta_{s'}\exp\left[\beta\sum_{\langle ss'\rangle}\theta_s\theta_{s'}\right]\\ &= \langle\theta_{0,0}\theta_{0,1}\rangle_{\text{Ising}}\\ &= \frac{1}{2\pi}\int_0^{2\pi} d\theta\left[\frac{(1-\alpha_1 e^{i\theta})(1-\alpha_2 e^{-i\theta})}{(1-\alpha_1 e^{-i\theta})(1-\alpha_2 e^{i\theta})}\right]^{\frac{1}{2}},\end{aligned} \tag{8}$$

where $\alpha_1 = e^{-2\beta}\tanh\beta$ and $\alpha_1 = e^{-2\beta}/\tanh\beta$. Thus, we can see that the model can be mapped to the Ising model, which is exactly solvable [25].

IV. CORRELATIONS IN THE LATTICE

In this section, we discuss the correlations in the lattices. Both the local spin-spin correlations and the global correlation between a single spin and the rest of the lattice are considered. We find that the local correlations are classical, and the quantum correlation emerges only when considering the whole lattice. In both cases, the mutual information signals the critical behavior.

A. Local spin-spin correlation

First, let us look at the correlation of two local spins. We need to get the reduced density matrix of two spins. The set $\{\frac{1}{2}\hat{\sigma}_i^\mu\hat{\sigma}_j^\nu\}$, where μ,ν take the values $0, 1, 2, 3$ and $\hat{\sigma}^0 = \mathbf{1}$, contains 16 matrices, and they form a complete orthonormal basis for 4×4 Hermitian matrices under the Hilbert-Schmidt inner product $(A, B)_{\text{H-S}} \equiv \text{tr}(A^\dagger B)$ [24]. Conveniently, the reduced density matrix can be written as an expansion in the basis $\{\frac{1}{2}\hat{\sigma}_i^\mu\hat{\sigma}_j^\nu\}$ [22,26],

$$\hat{\rho}_{ij} = \frac{1}{4}\sum_{\mu,\nu=0}^{3}\langle\hat{\sigma}_i^\mu\hat{\sigma}_j^\nu\rangle\hat{\sigma}_i^\mu\hat{\sigma}_j^\nu, \tag{9}$$

where $\langle\hat{\sigma}_i^\mu\hat{\sigma}_j^\nu\rangle = \text{Tr}(\hat{\rho}_{\text{g.s.}}\hat{\sigma}_i^\mu\hat{\sigma}_j^\nu) = \text{tr}(\hat{\rho}_{ij}\hat{\sigma}_i^\mu\hat{\sigma}_j^\nu)$ is the inner product of $\hat{\rho}_{ij}$ and $\hat{\sigma}_i^\mu\hat{\sigma}_j^\nu$.

Furthermore, most terms above can be eliminated because of the symmetry of the system. Draw a closed loop through the torus arbitrarily (as the red line shown in Fig. 1), and define a corresponding transformation $\hat{\mathcal{P}} = \prod_{\text{line}}\hat{\sigma}_i^z$. The Hamiltonian (4) is invariant under the transformation $\hat{\mathcal{P}}$. Also, $\hat{\rho}_{ij}$ should commute with any $\hat{\mathcal{P}}$. Only the terms $\mathbf{1}$, $\hat{\sigma}_i^z$, $\hat{\sigma}_j^z$ and $\hat{\sigma}_i^z\hat{\sigma}_j^z$ could exist, so we get

$$\begin{aligned}\hat{\rho}_{ij} &= \tfrac{1}{4}\left[\mathbf{1} + \langle\hat{\sigma}_i^z\rangle(\hat{\sigma}_i^z + \hat{\sigma}_j^z) + \langle\hat{\sigma}_i^z\hat{\sigma}_j^z\rangle\hat{\sigma}_i^z\hat{\sigma}_j^z\right]\\ &= \tfrac{1}{4}\left[\left(1+\langle\sigma_i^z\rangle\right) + \left(\langle\sigma_i^z\rangle + \langle\sigma_i^z\sigma_j^z\rangle\right)\sigma_i^z\right]\otimes\Pi_0\\ &\quad + \tfrac{1}{4}\left[\left(1-\langle\sigma_i^z\rangle\right) + \left(\langle\sigma_i^z\rangle - \langle\sigma_i^z\sigma_j^z\rangle\right)\sigma_i^z\right]\otimes\Pi_1.\end{aligned} \tag{10}$$

The density matrix is diagonal. It can be written in the form of $\hat{\rho}_{ij} = \sum p_n\rho_n\otimes\Pi_n$. According to what we have seen in Sec. II, the quantum discord of $\hat{\rho}_{ij}$ is zero. That means the correlations between any two local spins are always classical. This is quite different from other studies of quantum discord in the phase transition of 1D systems that are not topologically ordered [18–20], where the quantum discord of local spins shows different behavior in different phase areas and exhibits critical behavior.

This stems from the high symmetry of the topologically ordered system. This 2D system exhibits higher symmetry than other 1D Z_2-symmetric models [18,19]. The system is conserved under the transformation of $\hat{\mathcal{P}}$ along any closed loop, which eliminates all nondiagonal terms. So the quantum discord is zero in the topological phase area. It was stated in Ref. [2] that in a topologically ordered system, all observable properties should be invariant under smooth deformations (diffeomorphisms) of the space-time manifold, which means the only local operator that has nonvanishing correlation functions is the identity. For example, in the toric code model, only identity exists in the expansion Eq. (9) and $\hat{\rho}_{ij}\sim\mathbf{1}\otimes\mathbf{1}$, which means local spins are even uncorrelated. While the other phase area, where $\beta\to\infty$, is a fully magnetized phase, which obviously only contains classical information of probability. This is why local correlations are classical in both phases.

Besides, it was stated that a topological quantum phase transition cannot be described by the symmetry breaking of a local order parameter and involves a global rearrangement of nonlocal correlations [1]. However, recent research has indicated that some concepts in quantum information theory, even though they describe local properties, still signal the singularity in topological quantum phase transition [5,21,22]. Reduced fidelity and local magnetization were studied in the Castelnovo-Chamon model and the Kitaev toric code in a magnetic field, and they exhibit the critical behavior of the

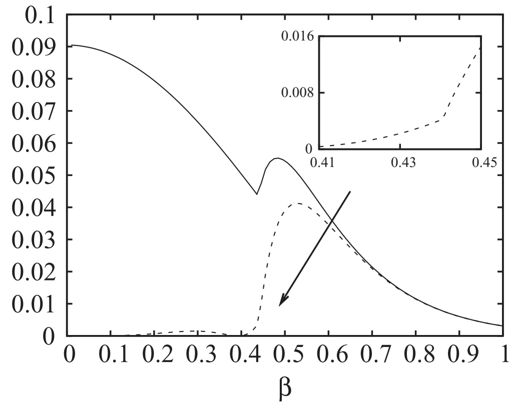

FIG. 2. Mutual information of two nearest spins $\sigma_i^z\sigma_{i''}^z$ (solid line) and $\sigma_i^z\sigma_{i'}^z$ (dashed line) as shown in Fig. 1. Critical change happens at $\beta_c = (1/2)\ln(\sqrt{2}+1)$, which is in accord with previous studies. The quantum discord of the two spins is always zero.

topological quantum phase transition. We calculate the mutual information $\mathcal{I}$ of the two nearest spins, which is also a local property (Fig. 2). The correlations in $\hat{\rho}_{ij}$ can be evaluated with the help of the mapping to Ising model, as mentioned in Sec. III. We can see that the mutual information of both nearest and next-nearest spins (in the dual lattice) exhibits critical behavior. But the next-nearest mutual information is much less sensitive.

B. Global correlation in the lattice

As we have seen in the last section, the correlations between local spins are completely classical in both phases. In this section, we calculate the correlation between a local spin and the rest of the whole lattice. As β increases, the system turns to the magnetic phase, and the correlation between a local spin and the lattice becomes more and more "classical."

To calculate the correlation of an arbitrary spin denoted by k with the lattice, we treat the rest of the lattice as a whole system. We can always rewrite the ground state as

$$\begin{aligned}|\text{g.s.}(\beta)\rangle &= \sum_x a_x|x\rangle|0\rangle_k + \sum_y b_y|y\rangle|1\rangle_k \\ &= a|X\rangle|0\rangle_k + b|Y\rangle|1\rangle_k,\end{aligned} \tag{11}$$

where $|X\rangle = \sum_x a_x|x\rangle$ and $|Y\rangle = \sum_y b_y|y\rangle$. The x, y in the basis vectors are the binary number representation of $g|0\rangle$ excluding the kth spin, as mentioned in Sec. III. Notice that $g|0\rangle$ and $g'|0\rangle (g \neq g')$ have at least four different spins. So we are sure that $|X\rangle$ and $|Y\rangle$ have no term in common, and $\langle X|Y\rangle = 0$. Therefore, we can treat $|\text{g.s.}(\beta)\rangle$ as a simple 2×2 entangled state. In this case, the quantum discord is equal to the entanglement of entropy [27,28],

$$D(\rho_{AB}) = \mathcal{I}(A:B)/2 = S(A) = S(B). \tag{12}$$

We calculate it in detail. The coefficients a_x, b_y are superposition coefficients in Eq. (5) correspondingly. So the value of $a^2(b^2)$ is just the Ising partition function with a constraint that $\sigma_k^z = 1(-1)$, in other words, $\theta_r\theta_{r'} = 1(-1)$, where r and r' are the nearest vertices of spin k.

$$\begin{aligned} a^2 &= \sum_{\{\theta_s\},\theta_r\theta_{r'}=1} \exp\left[\beta\sum_{\langle\theta_s\theta_{s'}\rangle}\theta_s\theta_{s'}\right] \Big/ Z(\beta), \\ b^2 &= \sum_{\{\theta_s\},\theta_r\theta_{r'}=-1} \exp\left[\beta\sum_{\langle\theta_s\theta_{s'}\rangle}\theta_s\theta_{s'}\right] \Big/ Z(\beta). \end{aligned} \tag{13}$$

Notice that $a^2 - b^2$ is just the nearest correlation function $\langle\theta_{0,0}\theta_{0,1}\rangle$.

$$\begin{aligned}\langle\theta_{0,0}\theta_{0,1}\rangle &= \left[\sum_{\theta_r\theta_{r'}=1} e^{\beta\sum\theta_s\theta_{s'}} - \sum_{\theta_r\theta_{r'}=-1} e^{\beta\sum\theta_s\theta_{s'}}\right] \Big/ Z(\beta) \\ &= a^2 - b^2. \end{aligned} \tag{14}$$

Together with $a^2 + b^2 = 1$, we can get the value of a, b.

Now we calculate the quantum discord of the ground state in Eq. (11), $\hat{\rho}_{\text{g.s.}} = |\text{g.s.}(\beta)\rangle\langle\text{g.s.}(\beta)|$. Instead of doing all the possible projective measurements to the spin, equivalently, we implement all possible local unitary operations on spin k and then measure it by $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$.

$$\begin{aligned}\Pi_0 U^\dagger \hat{\rho}_{\text{g.s.}} U \Pi_0 &= \begin{pmatrix} a^2\cos^2\frac{\theta}{2} & \frac{1}{2}ab\sin\theta e^{i\phi} \\ \frac{1}{2}ab\sin\theta e^{-i\phi} & b^2\sin^2\frac{\theta}{2}\end{pmatrix} \otimes |0\rangle\langle 0| \\ &= \tilde{\rho}_0 \otimes \Pi_0, \end{aligned}$$

$$\begin{aligned}\Pi_1 U^\dagger \hat{\rho}_{\text{g.s.}} U \Pi_1 &= \begin{pmatrix} a^2\sin^2\frac{\theta}{2} & -\frac{1}{2}ab\sin\theta e^{i\phi} \\ -\frac{1}{2}ab\sin\theta e^{-i\phi} & b^2\cos^2\frac{\theta}{2}\end{pmatrix} \otimes |1\rangle\langle 1| \\ &= \tilde{\rho}_1 \otimes \Pi_1, \end{aligned} \tag{15}$$

where

$$U = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\phi} \\ \sin\frac{\theta}{2}e^{i\phi} & \cos\frac{\theta}{2}\end{pmatrix}. \tag{16}$$

The unnormalized post-measurement density matrices $\tilde{\rho}_0$ and $\tilde{\rho}_1$ in Eq. (15) both have only one nonzero eigenvalue, respectively, i.e., $\lambda_k = [a^2 + b^2 + (-1)^k(a^2 - b^2)\cos\theta]/2$, where $k = 0, 1$. That means the conditional information about the lattice after the measurement of a local spin is zero,

$$S(\rho_{\text{g.s.}}|\{\Pi_i\}) = \sum_{k=0,1}\lambda_k S(\tilde{\rho}_k/\lambda_k) = 0. \tag{17}$$

Therefore, $\mathcal{J}$ is always equal to the entanglement of entropy S, no matter what measurement we impose on the local spin. So the quantum discord is equal to S,

$$\begin{aligned} D(\rho_{\text{g.s.}}) &= \mathcal{J} = \mathcal{I}/2 = S \\ &= -a^2\log_2 a^2 - b^2\log_2 b^2, \end{aligned} \tag{18}$$

where

$$\begin{aligned} a^2 &= (1 + \langle\theta_{0,0}\theta_{0,1}\rangle)/2, \\ b^2 &= (1 - \langle\theta_{0,0}\theta_{0,1}\rangle)/2. \end{aligned} \tag{19}$$

The correlation function $\langle\theta_{0,0}\theta_{0,1}\rangle$ has an explicit analytic form, Eq. (8) [25].

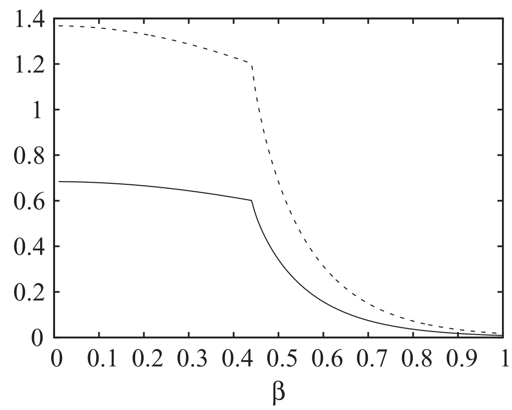

FIG. 3. Global correlations between a local spin and the rest of the whole lattice. Here, the quantum discord (solid line) is equal to the entropy of entanglement, and just one-half of the mutual information (dashed line) of the pairwise system. Both the quantum discord and the mutual information show critical change at the phase transition point.

The quantum discord and mutual information of the global correlation is shown in Fig. 3. Comparing it with that of local correlation, the quantum discord is not zero, which means the quantum correlation exists in the lattice globally. It also signals the critical point in the phase transition, just like the mutual information. With the increase of β, the quantum discord decreases to zero, which means the global quantum correlation disappears gradually.

In summary, the quantum correlation hides in the global lattice. We can only get classical correlations between local spins. All these results of correlations seem to suggest that the ground state of the topologically ordered system behaves as a generalized GHZ state. The quantum information is encoded in the lattice globally, and so it can be protected better than in other systems.

V. CONCLUSION

In this paper, we studied the correlations in the Castelnovo-Chamon model. Both local and global correlations were studied. The correlations were measured by quantum discord. As we have seen, local spins are classically correlated, although the Hamiltonian is very complicated, whereas the quantum correlation is hidden in the lattice globally. This is quite peculiar compared to previous studies. We found that these distinctive characters result from the high symmetry of the 2D topologically ordered system. The spins along any loop on the torus are Z_2 symmetric. This strict constraint clears the quantum correlations between local spins. Only global quantum correlation exists, just like a generalized GHZ state. We believe that this is a generic property in topological quantum phase transition because of the particular symmetry of topologically order systems, as mentioned previously.

Moreover, we calculated the mutual information of the two nearest spins, which signals critical behavior of the topological quantum phase transition. Similar to a previous study of fidelity, mutual information also works as a local probe of the topologically ordered phase, although topological order cannot be described by the symmetry-breaking of local order parameter. More study is required of correlations in other more realistic systems.

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