

Entropy dynamics of a dephasing model in a squeezed thermal bath

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We study the entropy dynamics of a dephasing model, where a two-level system (TLS) is coupled with a squeezed thermal bath via a nondemolition interaction. This model is exactly solvable, and the time-dependent states of both the TLS and its bath can be obtained exactly. Based on these states, we calculate the entropy dynamics of both the TLS and the bath, and find that the dephasing rate of the system relies on the squeezing phase of the bath. In the zero-temperature and high-temperature limits, both the system and bath entropy increase monotonically in the coarse-grained time scale. Moreover, we find that the dephasing rate of the system relies on the squeezing phase of the bath, and this phase dependence cannot be precisely derived from the Born-Markovian approximation which is widely adopted in open quantum systems.

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I. INTRODUCTION

When an open quantum system is coupled with a thermal reservoir, it turns out that the classical thermodynamics relations also apply [1–4]. However, current technology makes it possible to create a nonthermal environment for quantum systems; for example, quantum coherence or squeezing could also exist in the reservoir, and makes it a nonthermal bath [5–10]. In these cases, it is permissible that the conventional thermodynamics relations do not hold. Even more strikingly, a quantum heat engine working with such a nonthermal quantum bath could seemingly exceed the Carnot bound [5–7,9].

This is because indeed conventional thermodynamics only concerns thermal equilibrium reservoirs; particularly, the thermal entropy $dS = \delta Q/T$ is only defined for the equilibrium state. For nonthermal baths, the conventional entropy relations should be reconsidered. There are some different approaches dealing with such problems. For example, some external work should be considered to maintain the quantum coherence in the bath [3,4], excess heat should be taken into account [11], or the heat should be redefined with the help of a passive state [10,12].

Recently, it was noticed that the entropy production in conventional thermodynamics can be understood as the correlation generation between an open quantum system and its thermal reservoir [13–22]. For example, for a thermal state $\rho_B(0) = Z^{-1} \exp[-\hat{H}_B/T]$,¹ assuming the bath state does not change too much from the initial state, the von Neumann entropy of the bath state gives $\dot{S}_B = -\text{tr}[\dot{\rho}_B(t) \ln \rho_B(t)] \approx -\text{tr}[\dot{\rho}_B(t) \ln \rho_B(0)] = \frac{d}{dt} \langle \hat{H}_B \rangle / T$ [19,23]. If all the bath energy loss is gained by the system, $\delta \langle \hat{H}_B \rangle \approx -\delta \langle \hat{H}_S \rangle$, then it turns out that the informational entropy change of the bath \dot{S}_B is just equivalent to the thermal entropy $\delta Q/T$ of the system.

More importantly, in squeezed thermal baths, the conventional thermal entropy $dS = \delta Q/T$ does not apply [6]. But it turns out that the system-bath (S-B) correlation still increases monotonically [19]. Therefore, it is worthwhile to study the entropy dynamics of both the system and its bath in more examples, especially the exactly solvable models.

In this paper, we study the entropy dynamics in a dephasing model, where a two-level system (TLS) interacts with a squeezed thermal bath via nondemolition coupling. This model is well adopted to describe physical systems such as the exciton-phonon interaction, molecular oscillation, and photosynthesis process [24–26]. This model is exactly solvable, and plenty of studies have been done considering that the bath is a thermal equilibrium state [27–29]. Here we consider that the bath starts from a squeezed thermal state [30] and study the dynamics of both the system and the bath, especially their entropy.

We obtain the exact evolution of both the system and bath states, and find that both the system and bath entropy $S_{S,B}$ increase monotonically. Moreover, we find that the dephasing rate of the system relies on the squeezing phase θ of the bath. Particularly, in the high-temperature limit, the system dephasing process is Markovian, and the quantum coherence decays exponentially, with the dephasing rate $\kappa = 2\lambda T [\cosh 2r - \frac{\ln 4}{\pi} \sinh 2r \sin \delta\theta]$, where r is the squeezing strength, λ is a unitless number characterizing the S-B coupling strength, and $\delta\theta$ is the phase difference between the squeezing phase θ relative to the phase of the coupling strength. We also notice that this dephasing rate κ cannot be precisely obtained from the Born-Markovian approximation widely adopted in open quantum systems.

We arrange the paper as follows: in Sec. II, we introduce the dephasing model, and show how to get the exact evolution operator; in Sec. III, we study the dynamics of the system and its entropy, and discuss the cases of zero-temperature and high-temperature limits; in Sec. IV, we study the bath dynamics, and discuss how to calculate the bath entropy approximately;

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¹ $\hat{H}_{S(B)}$ is the Hamiltonian of the system or bath, and T is the temperature.

finally we draw our summary in Sec. V. Some of the calculation details are presented in Appendices.

II. DEPHASING MODEL IN A SQUEEZED BATH

Here we first introduce the dephasing model, which is composed of a TLS ($\hat{H}_S = \frac{1}{2}\Omega_0\hat{\sigma}^z$) coupled with a boson bath ($\hat{H}_B = \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k$) [27–29]. The system-bath interaction is described by the following nondemolition Hamiltonian,

$$\hat{V}_{SB} = \hat{\sigma}^z \cdot \left[\sum_k (g_k \hat{b}_k + g_k^* \hat{b}_k^\dagger) \right], \quad (1)$$

where $\hat{\sigma}^z = |e\rangle\langle e| - |g\rangle\langle g|$, and $|e\rangle$, $|g\rangle$ are the excited and ground states.

Notice that $[\hat{H}_S, \hat{V}_{SB}] = 0$; thus the system energy is always conserved, and the populations $p_{e,g}$ on $|e, g\rangle$ do not change with time. But the bath energy is not conserved since $[\hat{H}_B, \hat{V}_{SB}] \neq 0$. Therefore, unlike the discussions in conventional thermodynamics, this open system can never exchange energy with its thermal reservoir [31,32]. But it is still meaningful to discuss the information and correlation exchange between the system and the bath [19,26], as we will do below.

The evolution behavior of the total S-B system is exactly solvable [27–29]. In the interaction picture of $\hat{H}_S + \hat{H}_B$, it turns out that the evolution operator can be written down as a separable form $U_I(t) = |e\rangle\langle e| \otimes U_e + |g\rangle\langle g| \otimes U_g$, where

$$U_e = \prod_k \hat{D}_k^+, \quad U_g = \prod_k \hat{D}_k^-, \quad (2)$$

$$\hat{D}_k^\pm := \exp\{\pm[\alpha_k(t)\hat{b}_k^\dagger - \alpha_k^*(t)\hat{b}_k]\}.$$

Notice that $U_{e,g}$ are both products of \hat{D}_k^\pm , where \hat{D}_k^\pm is a displacement operator for mode \hat{b}_k , and we denote

$$\alpha_k(t) := \mu_k(1 - e^{i\omega_k t}), \quad \mu_k := g_k/\omega_k \quad (3)$$

for the amount of displacement (see also the derivation in Appendix A).

With this evolution operator, we can obtain the exact state $\rho_{SB}(t)$ at any time starting from $\rho_{SB}(0) = \rho_S(0) \otimes \rho_B(0)$ (hereafter all the density matrices are in the interaction picture). Plenty of studies have been done considering the initial state of the bath as a thermal equilibrium state [27,33,34]. In this paper we study the case in which the initial state of the boson bath is a squeezed thermal state [30]

$$\rho_B(0) = \hat{S}\rho_{th}\hat{S}^\dagger, \quad \rho_{th} = \frac{1}{Z}e^{-\frac{1}{T}\hat{H}_B}, \quad (4)$$

where T is the temperature for the thermal state ρ_{th} , Z is the normalization constant, $\hat{S} = \prod_k \hat{s}_k$ is the squeezing operator for the boson bath, and \hat{s}_k is the squeezing operator for the \hat{b}_k mode:

$$\hat{s}_k = \exp\left[\frac{1}{2}\xi_k^* \hat{b}_k^2 - \frac{1}{2}\xi_k (\hat{b}_k^\dagger)^2\right], \quad \xi_k = r_k e^{i\theta_k} \quad (r_k \geq 0). \quad (5)$$

Here r_k and θ_k indicate the squeezing strength and phase, respectively.

III. SYSTEM DYNAMICS

In this section, we study the dynamics of the open system $\rho_S(t) = \text{tr}_B[U_I(t)\rho_{SB}(0)U_I^\dagger(t)]$. Since the populations $p_{e,g}$ on $|e, g\rangle$ do not change with time, the density matrix of the open system $\rho_S(t)$ can be written as

$$\rho_S(t) = \begin{bmatrix} p_e & \rho_{eg}e^{-\Gamma(t)} \\ \rho_{ge}e^{-\Gamma(t)} & p_g \end{bmatrix}. \quad (6)$$

The time-dependent behavior of the off-diagonal terms shows as

$$e^{-\Gamma(t)} = \text{tr}_B[U_e \rho_B(0) U_g^\dagger] \\ = \text{tr}_B\left(\rho_B(0) \exp\left\{2 \sum_k [\alpha_k(t)\hat{b}_k^\dagger - \alpha_k^*(t)\hat{b}_k]\right\}\right). \quad (7)$$

If the decay factor $\Gamma(t)$ linearly depends on t , it means the quantum coherence terms decay exponentially and that is a Markovian process [27,35,36].

Notice that the above expression for $e^{-\Gamma(t)}$ is just the characteristic function for the Wigner representation of $\rho_B(0)$, which is a squeezed thermal state of all bath modes [28,35,37,38]. Thus we obtain

$$\Gamma(t) = \sum_k \frac{1}{2} |\gamma_k(t)|^2 \coth \frac{\omega_k}{2T}, \quad (8)$$

where $\gamma_k(t) := 2\alpha_k(t) \cosh r_k + 2\alpha_k^*(t) e^{i\theta_k} \sinh r_k$. Substituting $\alpha_k(t) := \frac{g_k}{\omega_k}(1 - e^{i\omega_k t})$ into the above expression, the decay factor becomes

$$\Gamma(t) = \sum_k \frac{4|g_k|^2}{\omega_k^2} (1 - \cos \omega_k t) \coth \frac{\omega_k}{2T} [\cosh 2r_k \\ - \sinh 2r_k \cos(\omega_k t - \Delta\theta_k)], \quad (9)$$

where $\Delta\theta_k := \theta_k - 2\phi_k$ is the phase difference between the squeezing phase θ_k relative to the phase of the coupling strength g_k ($g_k = |g_k|e^{i\phi_k}$, $\phi_k := \arg[g_k]$).

Now we introduce a coupling spectral density $J(\omega) := 2\pi \sum_k |g_k|^2 \delta(\omega - \omega_k)$ [27,39]; then the above summation can be written as an integral for continuous bath modes (considering that $r_k = r$, $\Delta\theta_k = \delta\theta$ are constants):

$$\Gamma(t) = \int_0^\infty \frac{d\omega}{2\pi} 4J(\omega) \coth \frac{\omega}{2T} \cdot \frac{1 - \cos \omega t}{\omega^2} \\ \times [\cosh 2r - \sinh 2r \cos(\omega t - \delta\theta)]. \quad (10)$$

Here we adopt the Ohmic coupling spectral density with an exponential cutoff (with cutoff frequency Ω_c), i.e., $J(\omega) = \lambda\omega e^{-\omega/\Omega_c}$, which could lead to a Markovian process in many cases [27,35,40]. Here λ is a unitless number indicating the coupling strength.

A. Zero-temperature case

When the temperature $T \rightarrow 0$, we have $\coth \frac{\omega}{2T} \rightarrow 1$. In this case, the initial state of the bath is a pure state squeezed from the vacuum. The decay factor of the open system can be integrated out, and that is

$$\Gamma(t) = \frac{\lambda}{\pi} [A_t \cosh 2r - \sinh 2r (B_t \cos \delta\theta + C_t \sin \delta\theta)], \quad (11)$$

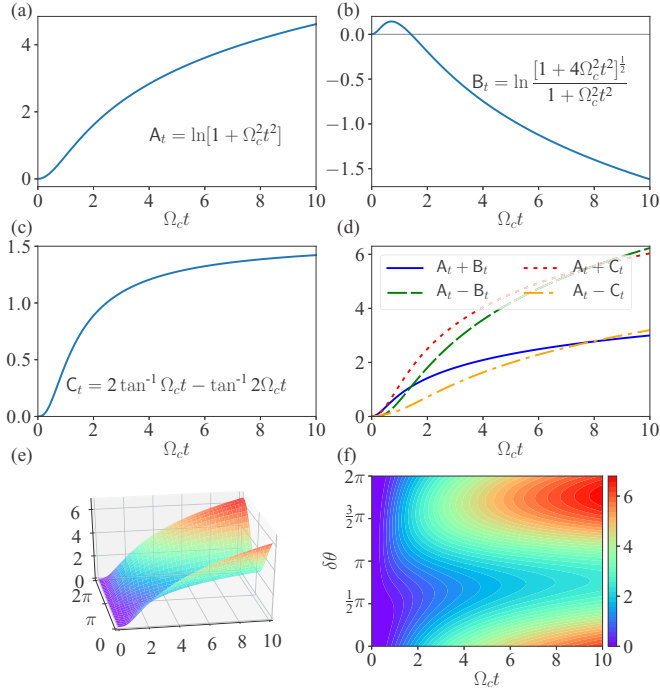


FIG. 1. The time dependence of the coefficients in zero temperature [Eq. (12)] (a) A_t , (b) B_t , (c) C_t , (d) $A_t \pm B_t$, $A_t \pm C_t$, and (e), (f) $A_t - (B_t \cos \delta\theta + C_t \sin \delta\theta)$.

where we denote ($\tau := \Omega_c t$)

$$\begin{aligned} A_t &= \ln[1 + \tau^2], & B_t &= \ln \frac{[1 + 4\tau^2]^{\frac{1}{2}}}{1 + \tau^2}, \\ C_t &= 2 \tan^{-1} \tau - \tan^{-1} 2\tau. \end{aligned} \quad (12)$$

It is simple to see that A_t and B_t both give rise to a power-law decay behavior. However, the factor C_t shows a quite different decay behavior. For very short time scale $t \lesssim \Omega_c^{-1}$, we have $C_t \approx 2\Omega_c^3 t^3$, which leads to a cubic exponential decay behavior $\sim \exp[-\frac{2}{\pi}\Omega_c^3 t^3]$. But for a long time scale $t \gg \Omega_c^{-1}$ ($\Omega_c t \gg 1$), approximately we have $\tan^{-1} \Omega_c t \approx \tan^{-1} 2\Omega_c t \approx \pi/2$, and thus $C_t \approx \pi/2$ is just a constant. This is quite different from the results of thermal baths [27].

We show the time-dependent coefficients A_t , B_t , and C_t in Fig. 1. By checking the positivity of $d\Gamma(t)/dt$ in the area $t \geq 0$, it is straightforward to prove that $\Gamma(t)$ is a monotonically increasing function for any squeezing parameters r and θ , which means the coherence of the TLS always decays monotonically (see Appendix C).

B. High-temperature limit

Now we consider the high-temperature limit. In this case, we have $\coth \frac{\omega}{2T} \approx \frac{2T}{\omega}$, and put it into the integral Eq. (10). However, the singularity in the denominator (ω^2) still makes it uneasy for integration. Here we eliminate this singularity by taking the derivative of t to the second order in the integral [27], and it turns out that $\partial_t^2 \Gamma(t)$ can be integrated out. Then the decay factor $\Gamma(t)$ can be obtained by integrating over t under the initial conditions $\Gamma|_{t=0}$ and $\partial_t \Gamma|_{t=0}$. Obviously, from Eq. (6) we know $\Gamma|_{t=0} = 0$. And $\partial_t \Gamma|_{t=0}$ can be obtained by

the integration of Eq. (10) by taking the derivative of t to the first order and then setting $t = 0$, which gives $\partial_t \Gamma|_{t=0} = 0$. It turns out that the decay factor $\Gamma(t)$ still has the same form as Eq. (11), but the time-dependent coefficients A_t , B_t , and C_t now become ($\tau := \Omega_c t$)

$$\begin{aligned} A_t &= \frac{2T}{\Omega_c} (2\tau \tan^{-1} \tau - \ln[1 + \tau^2]), \\ B_t &= \frac{2T}{\Omega_c} \left(2\tau [\tan^{-1} 2\tau - \tan^{-1} \tau] - \ln \frac{[1 + 4\tau^2]^{\frac{1}{2}}}{1 + \tau^2} \right), \\ C_t &= \frac{2T}{\Omega_c} \left([\tan^{-1} 2\tau - 2 \tan^{-1} \tau] + \tau \ln \frac{1 + 4\tau^2}{1 + \tau^2} \right). \end{aligned} \quad (13)$$

For the time scale $t \gg \Omega_c^{-1}$, considering $\Omega_c \gg T$, the above time-dependent factors become

$$A_t \approx 2\pi T t, \quad B_t \approx 0, \quad C_t \approx 2T t \ln 4. \quad (14)$$

Therefore, the decay factor $\Gamma(t)$ depends linearly on t , $\Gamma(t) = \kappa t$, where we define the decay rate as

$$\kappa := 2\lambda T \left[\cosh 2r - \frac{\ln 4}{\pi} \sinh 2r \sin \delta\theta \right]. \quad (15)$$

That means that the coherence of the TLS decays exponentially, and this is a Markovian process. Notice that Ω_c^{-1} is a very short time compared with the system dynamics, and $t \gg \Omega_c^{-1}$ just means after the relaxation time of the bath. When there is no squeezing, this result returns to the thermal bath result in previous studies [27,34].

Notice that here $\ln 4/\pi \approx 0.44 < 1$; thus the decay rate κ is always positive for any phase difference $\delta\theta$. It is worth noticing that the decay rate κ depends on the phase difference $\delta\theta$. Especially, when $\delta\theta = 3\pi/2$, we have $\sin \delta\theta = -1$, and thus the decay rate κ is suppressed; also, when $\delta\theta = \pi/2$, we have $\sin \delta\theta = 1$, and the decay rate κ gets the maximum enhancement. When the squeezing strength r is strong, $\cosh r \approx \sinh r \approx e^r/2$, and thus the decay rate is $\kappa \approx \lambda T e^r [1 - \sin \delta\theta (\ln 4/\pi)]$.

Using the Born-Markovian approximation [27], we can also derive a master equation describing the Markovian dephasing behavior (Appendix B), i.e.,

$$\dot{\rho}_S = \frac{1}{2} \kappa' ([\sigma_z \rho_S, \sigma_z] + [\sigma_z, \rho_S \sigma_z]). \quad (16)$$

But the decay rate is $\kappa' = 2\lambda T (\cosh 2r - \sinh 2r \cos \delta\theta)$. For thermal bath case ($r = 0$), this Born-Markovian dephasing rate κ' coincides with the one obtain from the exact evolution [27]; but for a squeezed thermal bath, the Born-Markovian master equation is not precise enough. We will discuss the reason for this inconsistency later.

C. Exact result and the system entropy

The exact result can be obtained by making the following expansion in the integral Eq. (10),

$$\coth \frac{\omega}{2T} = 1 + 2 \sum_{n=1}^{\infty} e^{-\frac{n\omega}{2T}}. \quad (17)$$

The zeroth order leads to the same integral as the above zero-temperature case, and the other terms give rise to integrals similar to those of the above high-temperature case,

except that the exponential cutoff should be corrected to be $\exp[-\frac{\omega}{\Omega_c} - \frac{n\omega}{2T}]$.

As a result, the decay factor still has the form of Eq. (11), but the coefficients A_t , B_t , and C_t are changed to be

$$\begin{aligned} A_t &= a_0 + 2 \sum_{n=1}^{\infty} a_n, & B_t &= b_0 + 2 \sum_{n=1}^{\infty} b_n, \\ C_t &= c_0 + 2 \sum_{n=1}^{\infty} c_n, \end{aligned} \quad (18)$$

where a_0 , b_0 , and c_0 are exactly the same with the coefficients A_t , B_t , and C_t in Eq. (12) (zero-temperature result), and a_n , b_n , and c_n have the same form with the coefficients A_t , B_t , and C_t in Eq. (13) (high-temperature result), except that the cutoff energy Ω_c in Eq. (13) should be corrected to be $\Omega_c \rightarrow \Omega_c/[1 + \frac{n\Omega_c}{2T}]$ correspondingly.

When the off-diagonal terms of the TLS decrease, the system entropy always increases. The TLS state can be always written as $\rho_s = \frac{1}{2}(\mathbf{1} + \sum v_i \hat{\sigma}^i)$, where $i = x, y, z$, and $v_i := \text{tr}[\rho_s \hat{\sigma}^i]$. Then the two eigenvalues of ρ_s are $\frac{1}{2}(1 \pm u)$, where $u = [v_x^2 + v_y^2 + v_z^2]^{\frac{1}{2}} \in [0, 1]$; thus the entropy of the TLS is

$$\begin{aligned} S_s &= \ln 2 - \frac{1}{2}[(1+u)\ln(1+u) + (1-u)\ln(1-u)], \\ \dot{S}_s &= -\frac{1}{2}\dot{u} \ln \frac{1+u}{1-u}. \end{aligned} \quad (19)$$

Notice that in this dephasing model, v_z does not change; thus $\dot{u} = (v_x \dot{v}_x + v_y \dot{v}_y)/u$ [33].

Therefore, when u decreases, the system entropy S_s increases. In both cases of the zero-temperature and high-temperature limits, the off-diagonal terms decay monotonically to zero, which indicates u decreases and the system entropy S_s increases monotonically.

IV. BATH DYNAMICS

In this section, we study the dynamics of the bath state, especially the entropy change of the bath.

Using the evolution operator $U_I(t)$ given in Sec. II, we can exactly write down the time-dependent bath state, i.e.,

$$\rho_B(t) = p_e \rho_B^+(t) + p_g \rho_B^-(t), \quad (20)$$

where $\rho_B^+(t) = U_e \rho_B(0) U_e^\dagger$ and $\rho_B^-(t) = U_g \rho_B(0) U_g^\dagger$.

Notice that the evolution operators $U_{e,g}$ [Eq. (2)] are both products of the displacement operators \hat{D}_k^\pm of each bath mode \hat{b}_k . Therefore, similarly to the initial state $\rho_B(0)$, $\rho_B(t)$ always keeps a product form $\rho_B(t) = \bigotimes_k \varrho_k(t)$, where

$$\varrho_k(t) = p_e \varrho_k^+(t) + p_g \varrho_k^-(t) \quad (21)$$

is the state of the bath mode \hat{b}_k , and $\varrho_k^\pm(t) = \hat{D}_k^\pm \varrho_k(0) [\hat{D}_k^\pm]^\dagger$. Thus the von Neumann entropy of $\rho_B(t)$ can be calculated by $\dot{S}[\rho_B(t)] = \sum_k \dot{S}[\varrho_k(t)]$.

The evolution of $\varrho_k^\pm(t)$ has a quite clear picture in the phase space of the Wigner function; i.e., they are displaced Gaussian packages with displacement $\pm\alpha_k(t)$. Initially, $\varrho_k(0)$ is a squeezed thermal state centered at the original point. The operators $U_{e,g}$ displace $\varrho_k(0)$ to the new centers at $\langle \hat{b}_k \rangle = \pm\alpha_k(t) = \pm\mu_k(1 - e^{i\omega_k t})$ correspondingly. With the

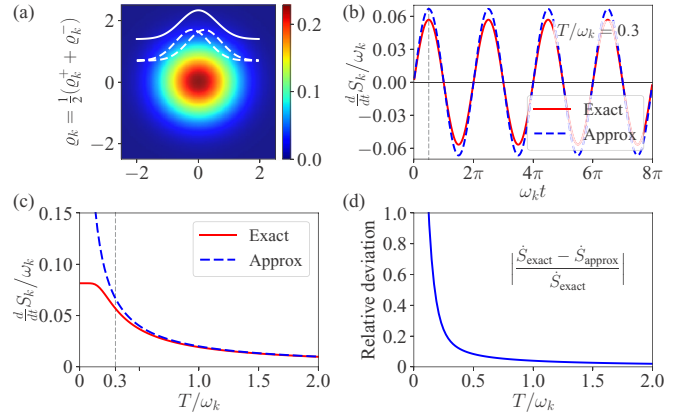


FIG. 2. (a) The Wigner function of a $\varrho_k = \frac{1}{2}(\varrho_k^+ + \varrho_k^-)$, where $\varrho_k^\pm = \hat{D}_k^\pm \varrho_{\text{th}}[\hat{D}_k^\pm]^\dagger$ are the displaced thermal states with displacement $\pm\alpha_k(t) = \pm\mu_k(1 - e^{i\omega_k t})$ and $\mu_k = 0.1$. (b) The time dependence of $\dot{S}[\varrho_k(t)]$ calculated by exact diagonalization (red solid) and the approximation Eq. (23) (blue dashed) for $T/\omega_k = 0.3$. (c) Comparison of the exact and approximated results for $\dot{S}[\varrho_k]_{\omega_k t = \frac{\pi}{2}}$ at different temperatures. (d) The relative deviation $|1 - \dot{S}_{\text{approx}}/\dot{S}_{\text{exact}}|$.

time going by, the package trajectories of $\varrho_k^\pm(t)$ form two cycles, and the state $\varrho_k(t)$ is their probabilistic mixture. That also means that indeed the bath never reaches any steady state.

As the result, the state $\varrho_k(t)$ of each bath mode is a non-Gaussian state. Thus it is still difficult to get the analytical result of its von Neumann entropy, although we know exactly the density matrix $\varrho_k(t)$ [41–43]. To bypass this difficulty, we calculate the dynamics of the bath state entropy with the following approximation [19,44]:

$$\dot{S}_B \approx -\text{tr}[\dot{\rho}_B(t) \ln \rho_B(t)] \approx -\text{tr}[\dot{\rho}_B(t) \ln \rho_B(0)], \quad (22)$$

assuming that the the state $\rho_B(t)$ does not change too much from $\rho_B(0)$ and thus $\ln \rho_B(t) \approx \ln \rho_B(0)$. This is quite similar to the idea of the Born approximation widely adopted in open quantum systems [27].

Under this approximation, for a thermal bath state $\rho_{\text{th}}(0) = Z^{-1} \exp(-\hat{H}_B/T)$, we obtain $\dot{S}_B \approx \frac{d}{dt} \langle \hat{H}_B \rangle / T$, which has the same form with the thermal entropy $dS = dQ/T$ in conventional thermodynamics [45]. Similarly, for a squeezed thermal bath, the bath entropy gives $\dot{S}_B \approx \frac{d}{dt} \langle \hat{S} \hat{H}_B \hat{S}^\dagger \rangle / T$ [6,19].

Thus, now the problem of the bath entropy dynamics is converted into calculating the dynamical variable expectations of the bath [19]. Since the exact evolution of the bath state $\rho_B(t)$ is known [Eq. (20)], we make a numerical comparison (Fig. 2) for the above approximation with the result calculated by exact diagonalization of the density matrix [for the single-mode state $\varrho_k(t) = p_e \varrho_k^+(t) + p_g \varrho_k^-(t)$].

When the two Gaussian packages $\varrho_k^\pm(t)$ are quite close to each other, their mixture $\varrho_k(t)$ well looks like a single Gaussian package [Fig. 2(a)]. The separation of $\varrho_k^\pm(t)$ is $2|\alpha_k(t)| = \frac{2|g_k|}{\omega_k} \sqrt{2 - 2 \cos \omega_k t} \leq 4|g_k|/\omega_k$, which is determined by the S-B interaction strength $|g_k|$; thus a weaker S-B interaction ($\lambda \ll 1$) makes a better approximation.

In Fig. 2, we consider an example of a thermal bath state, and the above approximation gives

$$\begin{aligned}\dot{S}[\varrho_k] &\approx \frac{\omega_k}{T} \frac{d}{dt} \langle \hat{b}_k^\dagger \hat{b}_k \rangle = \frac{\omega_k}{T} \frac{d}{dt} |\alpha_k(t)|^2 \\ &= \frac{\omega_k}{T} 2|\mu_k|^2 \omega_k \sin \omega_k t = \frac{2|g_k|^2}{T} \sin \omega_k t.\end{aligned}\quad (23)$$

This approximated result has the same oscillation behavior as the exact one [Fig. 2(b)], and the amplitudes also fit well (for $T/\omega_k = 0.3$). Thus we can use the maximum value (at $\omega_k t = \pi/2$) to characterize their deviation at different temperatures.

In Fig. 2(c) we show both the exact and approximated result for $\dot{S}[\varrho_k]_{\omega_k t = \pi/2}$ at different temperatures, as well as their relative deviation in Fig. 2(d). It turns out that indeed this approximation works well in the high-temperature regime, but not so well when $T \rightarrow 0$. Indeed, in the approximation (23), it is explicit to see that $\dot{S}[\varrho_k]$ diverges at low temperature, and this similar divergence behavior also appears in the conventional thermal entropy dQ/T [46]. But the exact result for the von Neumann entropy $\dot{S}[\varrho_k]$ does not diverge at low temperature [red solid line in Fig. 2(c)].

This is because the bath entropy $S[\varrho_k(t)]$ comes from two origins: one is the uncertainty due to finite temperature; the other one comes from the mixture proportion of $\varrho_k^\pm(t)$ encoded in the initial state probabilities $p_{e,g}$. From Eq. (23), we see that this approximated result does not depend on the probabilities $p_{e,g}$, which means this part of the uncertainty is omitted, and only the thermal fluctuation is counted.

In the high-temperature regime, the entropy of thermal fluctuation dominates and thus the approximation works well. In the low-temperature regime, the thermal uncertainty approaches zero; thus the non-Gaussian property of $\varrho_k(t)$ becomes important, and the approximation is not good. Therefore, the validity of the approximation (22) relies on the specific model and conditions. In this dephasing model, the TLS brings in nonlinearity to the model, which gives rise to the non-Gaussian property of the bath state. In this case, the above approximation does not work well.

Here we emphasize that $\dot{S}_B \approx \frac{d}{dt} \langle \hat{H}_B \rangle / T$ describes the bath entropy dynamics, although it has an analogous form to the thermal entropy $dS = dQ/T$, which is defined for the open system but not the bath. Thus, the above failure of the approximation in the low-temperature regime is not in conflict with the conventional thermodynamics.

For the squeezed thermal bath case, in the high-temperature regime, the above ‘‘semi-Born’’ approximation gives the entropy of one bath mode as

$$\begin{aligned}\dot{S}[\varrho_k(t)] &\approx \frac{\omega_k}{T} \frac{d}{dt} \langle \hat{s}_k \hat{b}_k^\dagger \hat{b}_k \hat{s}_k^\dagger \rangle \\ &= \frac{2|g_k|^2}{T} \{ \cosh 2r_k \sin \omega_k t - \sinh 2r_k \\ &\quad \times [\sin(2\omega_k t - \Delta\theta_k) - \sin(\omega_k t - \Delta\theta_k)] \},\end{aligned}\quad (24)$$

where $\Delta\theta_k = \theta_k - 2\phi_k$, and $\phi_k = \arg[g_k]$ is the phase of the coupling strength g_k . It is worth noticing that, similarly to the system dynamics [Eq. (15)], the entropy changing rate $\dot{S}[\varrho_k]$ depends on the squeezing phase $\Delta\theta_k$ of the bath mode.

This can be intuitively understood from Figs. 3(a) and 3(b), where the Wigner functions of $\varrho_k(t) = p_e \varrho_k^+ + p_g \varrho_k^-$

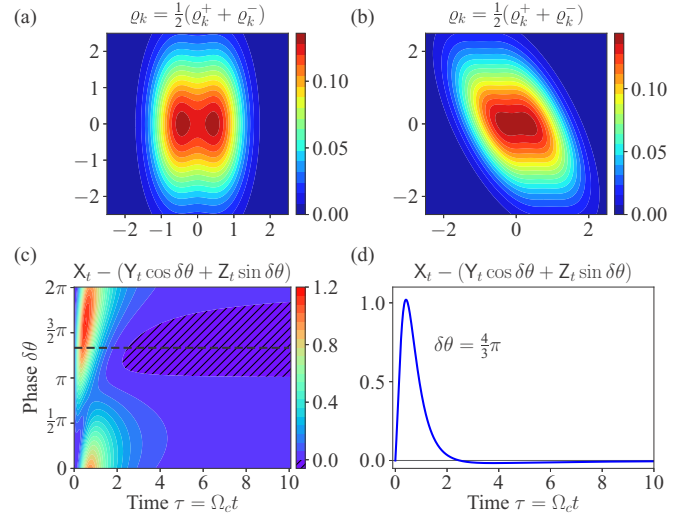


FIG. 3. (a), (b) The Wigner function for $\varrho_k = \frac{1}{2}(\varrho_k^+ + \varrho_k^-)$ at the maximum separation of ϱ_k^\pm (when $\omega_k t = \pi/2$). The squeezing phases θ are different in these two figures. (c), (d) $X_t - (Y_t \cos \delta\theta + Z_t \sin \delta\theta)$. The shadowed area indicates the bath entropy could decrease in this area, but the decreasing rate is quite small.

with different squeezing phases are shown at the maximum separation of ϱ_k^\pm ($\omega_k t = \pi/2$). Obviously, due to the different squeezing phases, the states $\varrho_k(t)$ differ a lot, and this phase dependence is also reflected in the expectation values $\langle \hat{b}_k^\dagger \hat{b}_k \rangle$, $\langle \hat{b}_k^2 \rangle$, and the entropy change $\dot{S}[\varrho_k(t)]$. Besides, it is clear to see that this difference cannot be eliminated by doing any phase rotation on the initial state.

This also explains why the dephasing rate of the system depends on the squeezing phase of the bath [Eq. (15)]. Since only the initial state of the bath is concerned when deriving the Born-Markovian master equation (Appendix B), the above bath dynamics is not taken into consideration; therefore, the dephasing rate κ' [Eq. (16)] derived from the Born-Markovian approximation does not coincide with the one obtained from the exact evolution.

Summing up $\dot{S}[\varrho_k]$ for all bath modes, the entropy changing rate of the total bath is given by the following integral:

$$\begin{aligned}\dot{S}_B &= \sum_k \dot{S}[\varrho_k] = \frac{2}{T} \int_0^\infty \frac{d\omega}{2\pi} J(\omega) \{ \cosh 2r \sin \omega t - \sinh 2r \\ &\quad \times [\sin(2\omega t - \delta\theta) - \sin(\omega t - \delta\theta)] \}.\end{aligned}\quad (25)$$

If we choose the Ohmic coupling spectral density $J(\omega) = \lambda\omega e^{-\omega/\Omega_c}$ as before, the above integral gives

$$\dot{S}_B = \frac{\lambda\Omega_c^2}{\pi T} [X_t \cosh 2r - \sinh 2r (Y_t \cos \delta\theta + Z_t \sin \delta\theta)],$$

where the coefficients are (denoting $\tau := \Omega_c t$)

$$\begin{aligned}X_t &= \frac{2\tau}{(1+\tau^2)^2}, \quad Y_t = \frac{2\tau(1-4\tau^2-14\tau^4)}{(1+\tau^2)^2(1+4\tau^2)^2}, \\ Z_t &= \frac{3\tau^2(3+5\tau^2-4\tau^4)}{(1+\tau^2)^2(1+4\tau^2)^2}.\end{aligned}\quad (26)$$

When there is no squeezing in the bath ($r = 0$), $\dot{S}_B = \frac{\lambda\Omega_c^2}{\pi T} X_t$, which is always positive for $t > 0$, meaning the bath entropy

increases monotonically. The terms with Y_t and Z_t can either increase or decrease with time, depending on the squeezing phase $\delta\theta$, but for practical squeezing parameters in current experiments, usually $\sinh 2r \ll \cosh 2r$; thus still the first increasing term X_t dominates, and \dot{S}_B remains positive.

In the strong-squeezing limit ($r \gg 1$), $\dot{S}_B \lesssim \frac{\lambda\Omega_c^2}{\pi T} \cosh 2r \cdot f(t, \delta\theta)$, where $f(t, \delta\theta) := X_t - (Y_t \cos \delta\theta + Z_t \sin \delta\theta)$ [see Figs. 3(c), 3(d)]. In most areas, $f(t, \delta\theta)$ remains positive, but for a certain phase $\delta\theta$ [the shadowed area in Fig. 3(c)], $f(t, \delta\theta)$ could become a small but negative value, indicating the decreasing of the bath entropy in this area. However, we also should notice that the time scale in Figs. 3(c), 3(d) is around $t \sim \Omega_c^{-1}$, which is very short compared with the system time scale ($\sim \kappa^{-1}$), and this is just the relaxation time of the bath. In the coarse-grained time scale (Markovian approximation), this decreasing of the bath entropy is negligible.

For the zero-temperature case, the total S-B system always stays in a pure state. As the time evolves, $\rho_{SB}(t)$ becomes a pure entangled state, and thus we always have $\dot{S}_B(t) = \dot{S}_S(t)$, which also increases monotonically, as already discussed in Sec. III. In this case, the thermal fluctuation does not contribute to the bath entropy, and \dot{S}_B comes entirely from the correlation with the TLS.

V. SUMMARY

In this paper, we study the entropy dynamics of a dephasing model, where a TLS is coupled with a squeezed thermal bath via a nondemolition interaction. We show the exact evolution operator, and the time-dependent states of both the TLS and its bath can be obtained exactly. Based on these states, we calculate the entropy dynamics of both the TLS and the bath, and find that the dephasing rate of the system relies on the squeezing phase of the bath. In the zero-temperature and high-temperature limits, both the system and bath entropy increase monotonically in the coarse-grained time scale (Markovian approximation).

Moreover, we find that the dephasing rate of the system relies on the squeezing phase θ of the bath. Particularly, in the high-temperature limit, the system dephasing process is Markovian, and the dephasing rate $\kappa = 2\lambda T[\cosh 2r - \frac{\ln 4}{\pi} \sinh 2r \sin \delta\theta]$. We also notice that this dephasing rate cannot be precisely given by the Born-Markovian approximation which is widely adopted in open quantum systems.

We also discuss the validity of using the thermal entropy analogy $\dot{S}_B \approx \frac{d}{dt} \langle \hat{H}_B \rangle / T$ to approximately calculate the bath entropy. For this dephasing model, when the bath temperature is high, the thermal fluctuation dominates the bath entropy dynamics, and this approximation works well; in the low-temperature regime, the non-Gaussian property of the bath state becomes more important, and this approximation does not work well.

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APPENDIX A: EVOLUTION OPERATOR

Here we show the derivation of the evolution operator $U_I(t)$ [Eq. (2)] [29]. In the interaction picture, the evolution operator can be written as the following time-ordered form:

$$\begin{aligned} U_I(t) &= \mathbf{T} \exp \left[-i \int_0^t ds \hat{V}_{SB}(s) \right] \\ &= \mathbf{T} \lim_{N \rightarrow \infty} \exp \left[-i \sum_{n=0}^{N-1} \hat{V}_{SB}(t_n) \delta t \right], \end{aligned} \quad (\text{A1})$$

where $\delta t = t/N$ is the time interval, $t_n = n\delta t$, and $\hat{V}_{SB}(t)$ is in the interaction picture.

For the interaction Hamiltonian (1) in the above dephasing model, we have $[\hat{V}_{SB}(t), \hat{V}_{SB}(s)] = 2i \sum_k |g_k|^2 \sin \omega_k(s-t)$. This is a c-number, thus we can use the Baker-Campbell-Hausdorff (BCH) formula:

$$e^{\sum_{n=1}^N A_n} = e^{A_1} e^{A_2} \dots e^{A_N} e^{-\frac{1}{2} \sum_{m < n} [A_m, A_n]}. \quad (\text{A2})$$

Then we obtain

$$U_I(t) = \mathbf{T} \lim_{N \rightarrow \infty} \prod_{n=0}^{N-1} e^{-i \hat{V}_{SB}(t_n) \delta t} \cdot e^{\frac{\delta t^2}{2} \sum_{m < n} [\hat{V}_{SB}(t_m), \hat{V}_{SB}(t_n)]}.$$

The above product is already time-ordered; thus the time-order operator \mathbf{T} can be removed. Then using the BCH formula in reverse, the evolution operator becomes

$$\begin{aligned} U_I(t) &= \lim_{N \rightarrow \infty} \exp \left[-i \sum_{n=0}^{N-1} \hat{V}_{SB}(t_n) \delta t \right] = \exp \left[-i \int_0^t ds \hat{V}_{SB}(s) \right] \\ &= \exp \{ \hat{\sigma}^z \cdot [\alpha_k(t) b_k^\dagger - \alpha_k^*(t) b_k] \}, \end{aligned} \quad (\text{A3})$$

where $\alpha_k(t) = \mu_k(1 - e^{i\omega_k t})$ and $\mu_k = g_k/\omega_k$.

APPENDIX B: MARKOVIAN MASTER EQUATION

Here we use the Born-Markovian approximation to derive a master equation for the TLS [27,39]. The master equation is derived from

$$\dot{\rho}_S = -\text{tr}_B \int_0^\infty ds [\hat{V}_{SB}(t), [\hat{V}_{SB}(t-s), \rho_S(t) \otimes \rho_B(0)]], \quad (\text{B1})$$

where $\hat{V}_{SB}(t) = \hat{\sigma}^z \cdot (\sum_k g_k \hat{b}_k e^{-i\omega_k t} + g_k^* \hat{b}_k^\dagger e^{i\omega_k t}) := \hat{\sigma}^z \cdot \hat{B}(t)$. The above commutator gives

$$\begin{aligned} & \int_0^\infty ds \langle \hat{B}^\dagger(t) \hat{B}(t-s) \rangle [\hat{\sigma}^z \rho_S(t), \hat{\sigma}^z] \\ &= \sum_k \int_0^\infty ds \langle (g_k^* \hat{b}_k^\dagger e^{i\omega_k t} + g_k \hat{b}_k e^{-i\omega_k t}) \\ & \quad \times (g_k \hat{b}_k e^{-i\omega_k(t-s)} + g_k^* \hat{b}_k^\dagger e^{i\omega_k(t-s)}) \rangle \cdot [\hat{\sigma}^z \rho, \hat{\sigma}^z] \\ &= \int_0^\infty ds \int_0^\infty \frac{d\omega}{2\pi} J(\omega) (\{\tilde{n}(\omega) e^{i\omega s} + [\tilde{n}(\omega) + 1] e^{-i\omega s}\} \\ & \quad + [\tilde{u}(\omega) e^{-2i\omega t + i\omega s} + \text{H.c.}]) [\sigma_z \rho, \sigma_z], \end{aligned}$$

where we denote

$$\begin{aligned}\tilde{n}(\omega) &= \cosh 2r \left[\bar{n}_p(\omega) + \frac{1}{2} \right] - \frac{1}{2}, \\ \tilde{u}(\omega) &= -e^{i\delta\theta} \sinh 2r \left[\bar{n}_p(\omega) + \frac{1}{2} \right],\end{aligned}\quad (\text{B2})$$

and $\bar{n}_p(\omega) = [\exp(\omega/T) - 1]^{-1}$ is the Planck function.

Utilizing the formula

$$\int_0^\infty ds e^{i(\varepsilon - \omega)s} = \pi \delta(\varepsilon - \omega) + i\mathbf{P} \frac{1}{\varepsilon - \omega}, \quad (\text{B3})$$

we obtain (omitting the principle integral)

$$\begin{aligned}& \int_0^\infty ds \langle \hat{B}^\dagger(t) \hat{B}(t-s) \rangle \\ &= \lim_{\omega \rightarrow 0} J(\omega) \left\{ \left[\tilde{n}(\omega) + \frac{1}{2} \right] + \frac{1}{2} [\tilde{u}(\omega) e^{-2i\omega t} + \text{H.c.}] \right\}.\end{aligned}\quad (\text{B4})$$

Adopting the Ohmic spectrum $J(\omega) = \lambda\omega e^{-\frac{\omega}{\Omega_c}}$, we have

$$\lim_{\omega \rightarrow 0} \lambda\omega e^{-\frac{\omega}{\Omega_c}} \left[\bar{n}_p(\omega) + \frac{1}{2} \right] = \lambda T. \quad (\text{B5})$$

Now we obtain the master equation $\dot{\rho}_S = \mathcal{L}[\rho_S]$ where

$$\begin{aligned}\mathcal{L}[\rho_S] &= \frac{1}{2} \kappa' ([\sigma_z \rho_S, \sigma_z] + [\sigma_z, \rho_S \sigma_z]), \\ \kappa' &= 2\lambda T (\cosh 2r - \sinh 2r \cos \delta\theta).\end{aligned}\quad (\text{B6})$$

Notice that here the phase dependence in the dephasing rate κ' is different from what we obtained in the main text using the exact evolution operator, which is more precise.

APPENDIX C: THE MONOTONIC INCREASE OF THE DECAY RATE

Here we show the proof for the monotonic increase of the system decay factor $\Gamma(t)$ in Sec. III A; namely, $\dot{\Gamma}(t)$ is always positive. Since for any $r > 0$, we have $\cosh 2r > \sinh 2r > 0$; thus

$$\begin{aligned}\dot{\Gamma}(t) &= \frac{\pi}{\lambda \cosh 2r} \\ &\geq \dot{A}_t - \sqrt{\dot{B}_t^2 + \dot{C}_t^2} \left(\frac{\dot{B}_t}{\sqrt{\dot{B}_t^2 + \dot{C}_t^2}} \cos \delta\theta + \frac{\dot{C}_t}{\sqrt{\dot{B}_t^2 + \dot{C}_t^2}} \sin \delta\theta \right) \\ &= \dot{A}_t - \sqrt{\dot{B}_t^2 + \dot{C}_t^2} \cos(\delta\theta - \varphi) \\ &\geq \dot{A}_t - \sqrt{\dot{B}_t^2 + \dot{C}_t^2},\end{aligned}\quad (\text{C1})$$

where we denote $\varphi := \tan^{-1}(\dot{C}_t/\dot{B}_t)$ and $\tau := \Omega_c t$. Using the expression of A_t , B_t , and C_t [Eq. (12)], we have

$$\dot{A}_t - \sqrt{\dot{B}_t^2 + \dot{C}_t^2} = \frac{2\tau}{1 + \tau^2} \left(1 - \sqrt{\frac{1 + \tau^2}{1 + 4\tau^2}} \right) \geq 0. \quad (\text{C2})$$

Thus, in the area $\tau \geq 0$, we always have $\dot{\Gamma}(t) \geq 0$. Therefore, $\Gamma(t)$ increases monotonically, and is always positive [since $\Gamma(0) = 0$].

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