

## Production rate of the system-bath mutual information

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When an open system comes into contact with several thermal baths, the entropy produced by the irreversible processes ( $dS_i = dS - \sum_{\alpha} dQ_{\alpha}/T_{\alpha}$ ) keeps increasing, and this entropy production rate is always non-negative. However, when the system comes into contact with nonthermal baths containing quantum coherence or squeezing, this entropy production formula does not apply. In this paper, we study the increasing rate of mutual information between an open system and its environment. In the case of canonical thermal baths, we prove that this mutual information production rate could return exactly to the previous entropy production rate. Furthermore, we study an example of a single boson mode that comes into contact with multiple squeezed thermal baths, where the conventional entropy production rate does not apply, and we find that this mutual information production rate remains non-negative, which indicates a monotonic increase in the correlation between the system and its environment.

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### I. INTRODUCTION

The entropy change of a system can be considered to come from two origins, i.e.,  $dS = dS_e + dS_i$  [1–4], where  $dS_e$  comes from the exchange with external sources, and it could be either positive or negative;  $dS_i$  is the entropy change due to the irreversible processes. Then the second law is simply stated as  $dS_i/dt \geq 0$ , which means the entropy produced by the irreversible processes always increases, and  $R_{\text{ep}} := dS_i/dt$  is called the *entropy production rate* (EPr).

When the system comes into contact with a thermal bath with temperature  $T$ , we have  $dS_e = dQ/T$  (hereafter we refer to this as the *thermal entropy*), where  $dQ$  is the heat flowing into the system. If we have multiple independent thermal baths with different temperatures  $T_{\alpha}$  (Fig. 1), the EPr becomes

$$\frac{dS_i}{dt} = \frac{dS}{dt} - \sum_{\alpha} \frac{1}{T_{\alpha}} \frac{dQ_{\alpha}}{dt} := R_{\text{ep}}, \quad (1)$$

where  $dQ_{\alpha}$  is the heat coming from bath- $\alpha$  [1,4].

Furthermore, when an open quantum system is weakly coupled with multiple thermal baths, its dynamics can usually be described by the following Lindblad (GKSL) equation [5,6]:

$$\dot{\rho} = i[\rho, \hat{H}_S] + \sum_{\alpha} \mathcal{L}_{\alpha}[\rho], \quad (2)$$

where  $\mathcal{L}_{\alpha}[\rho]$  describes the dissipation due to bath- $\alpha$ . Utilizing  $\dot{S}[\rho] = -\text{tr}[\dot{\rho} \ln \rho]$  and  $\dot{Q}_{\alpha} = \text{tr}[\hat{H}_S \cdot \mathcal{L}_{\alpha}[\rho]]$ , the EPr (1) can be rewritten as the following Spohn formula (denoted as  $R_{\text{Sp}}$  hereafter) [7–12]:

$$R_{\text{ep}} = \sum_{\alpha} \text{tr}[\mathcal{L}_{\alpha}[\rho](\ln \rho_{\text{ss}}^{(\alpha)} - \ln \rho)] := R_{\text{Sp}}. \quad (3)$$

Here we call  $\rho_{\text{ss}}^{(\alpha)} = Z_{\alpha}^{-1} \exp[-\hat{H}_S/T_{\alpha}]$  the *partial steady state* associated with bath- $\alpha$ , satisfying  $\mathcal{L}_{\alpha}[\rho_{\text{ss}}^{(\alpha)}] = 0$ . It can be proved that  $R_{\text{Sp}} \geq 0$ , which means the irreversible entropy production keeps increasing (see the proof in Appendix A or Refs. [7,8]).

However, in the above discussion, the thermal entropy  $dS_e = dQ/T$  only applies for canonical thermal baths. If the bath is some noncanonical state containing quantum coherence or squeezing [13–15], the temperature is not well defined, thus

it is no longer proper to use  $dQ/T$  for  $dS_e$  [16], and the relations  $R_{\text{ep}} = R_{\text{Sp}}$  or  $R_{\text{ep}} \geq 0$  no longer hold either.

Therefore, for such nonthermal baths, the conventional thermodynamic description of the EPr does not apply, and it is believed that the corrections on work [17–19], or excess heat [16], should be considered in these baths.

Here we replace the thermal entropy term  $-dQ_{\alpha}/T_{\alpha}$  by the von Neumann entropy of bath- $\alpha$ ,  $\dot{S}_{B\alpha} = -\text{tr}[\dot{\rho}_{B\alpha} \ln \rho_{B\alpha}]$ . Furthermore, we assume that the multiple baths are independent from each other, thus we obtain  $\sum_{\alpha} \dot{S}_{B\alpha} = \dot{S}_B$ . Then this generalization becomes

$$R_{\mathcal{I}} = \frac{dS_S}{dt} + \frac{dS_B}{dt} = \frac{d}{dt}(S_S + S_B - S_{SB}) = \frac{d}{dt} \mathcal{I}_{SB}. \quad (4)$$

Here  $\dot{S}_{SB} = 0$  since the total system  $S + B$  evolves unitarily,<sup>1</sup> and  $\mathcal{I}_{SB} := S_S + S_B - S_{SB}$  is just the mutual information between the system and its environment, which measures their correlation [20–25]. Therefore, we call  $R_{\mathcal{I}}$  the *mutual information production rate* (MIPr).

$R_{\mathcal{I}}$  has a clear physical meaning: a positive  $R_{\mathcal{I}}$  indicates that the correlation between the system and its environment is increasing. In the following, we are going to show that this MIPr (4) indeed has quite a close connection with the previous EPr (1). For thermal baths of the open system, we can prove that this MIPr could return exactly to the conventional thermodynamic description of the EPr in the weak-coupling limit, namely  $R_{\mathcal{I}} = R_{\text{ep}}$ . That means, for a thermal bath, that conventional entropy production can be equivalently interpreted as mutual information production, and the second law statement ( $R_{\mathcal{I}} = R_{\text{ep}} \geq 0$ ) can also be understood as the system-bath correlation always keeps increasing.

Furthermore, we will study an example of a single boson contacted with multiple squeezed thermal baths. In this case, the conventional EPr does not apply. We calculate the MIPr under the weak-coupling limit and the Markovian

<sup>1</sup> $\dot{S}[\rho_{SB}] = -\text{tr}[\dot{\rho}_{SB} \ln \rho_{SB}] = -i \text{tr}[\rho_{SB} \hat{\mathcal{H}} \ln \rho_{SB} - \hat{\mathcal{H}} \rho_{SB} \ln \rho_{SB}] = 0$ , where  $\hat{\mathcal{H}}$  is the Hamiltonian of the total  $S + B$  system.

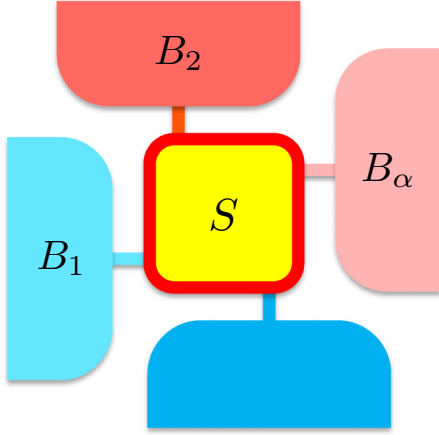


FIG. 1. Demonstration for an open quantum system ( $S$ ) interacting with its environment composed of multiple baths ( $B_\alpha$ ). The baths are independent from each other, and they do not have to be canonical thermal states.

approximation, and we find that it is exactly equal to the Spohn formula for nonthermal baths, thus we can prove  $R_{\mathcal{I}} \geq 0$ , which means that the monotonic increase of the system-bath correlation also holds in this squeezed bath example.

## II. MUTUAL INFORMATION PRODUCTION IN THERMAL BATHS

We consider first a system that is coupled with several thermal baths. In this case, the initial state of bath- $\alpha$  is  $\rho_{B\alpha}(0) = \mathcal{Z}_\alpha^{-1} \exp[-\hat{H}_{B\alpha}/T_\alpha]$ . Assuming  $\rho_{B\alpha}(t)$  does not change too much during evolution [26–28], we have  $\ln \rho_{B\alpha}(t) = \ln[\rho_{B\alpha}(0) + \delta\rho_t] \simeq \ln[\rho_{B\alpha}(0)] + o(\delta\rho_t)$ , thus the entropy change of bath- $\alpha$  is

$$\begin{aligned} \dot{S}_{B\alpha} &= -\text{tr}[\dot{\rho}_{B\alpha}(t) \ln \rho_{B\alpha}(t)] \simeq -\text{tr} \left[ \dot{\rho}_{B\alpha}(t) \ln \frac{e^{-\frac{\hat{H}_{B\alpha}}{T_\alpha}}}{\mathcal{Z}_\alpha} \right] \\ &= \frac{1}{T_\alpha} \frac{d}{dt} \langle \hat{H}_{B\alpha} \rangle \simeq -\frac{\dot{Q}_\alpha}{T_\alpha}. \end{aligned} \quad (5)$$

Here  $-\frac{d}{dt} \langle \hat{H}_{B\alpha} \rangle$  is the energy loss of bath- $\alpha$ , while  $\dot{Q}_\alpha$  is the energy gain of the system from bath- $\alpha$ , and they are equal to each other in the weak-coupling limit. Assuming the baths are independent from each other,  $\rho_B(t) \simeq \prod_\alpha \rho_{B\alpha}(t)$ , the MIPr becomes

$$R_{\mathcal{I}} = \dot{S}_S + \sum_\alpha \dot{S}_{B\alpha} = \dot{S}_S - \sum_\alpha \frac{\dot{Q}_\alpha}{T_\alpha} = R_{\text{ep}}. \quad (6)$$

Therefore, for thermal baths, the MIPr (4) is equal to the conventional thermodynamic description of the EPr (1).

Thus, the second law statement  $R_{\text{ep}} \geq 0$  is equivalent to  $R_{\mathcal{I}} \geq 0$ , which means the mutual information between the system and its environment continues to increase monotonically. This can be understood as an equivalent statement for the entropy production when the baths are canonical thermal baths. We notice that this equivalence was also shown in the ‘‘correlation entropy’’ approach [21,22,25].

## III. MUTUAL INFORMATION PRODUCTION IN SQUEEZED BATHS

Now we study an example of a single boson mode interacting with multiple squeezed thermal baths [11,14,15]. In this case, the thermal entropy  $dS_e = dQ/T$  cannot be used, and nor can the EPr (1). Here we calculate the MIPr (4), and we will prove that it is equal to the Spohn formula for nonthermal baths, thus it could remain non-negative,  $R_{\mathcal{I}} \geq 0$ .

### A. Master equation and Spohn formula

The Hamiltonians of the single boson mode and the bosonic bath are  $\hat{H}_S = \Omega \hat{a}^\dagger \hat{a}$ ,  $\hat{H}_B = \sum_\alpha \hat{H}_{B\alpha}$ , and  $\hat{H}_{B\alpha} = \sum_k \omega_{\alpha k} \hat{b}_{\alpha k}^\dagger \hat{b}_{\alpha k}$ , and they interact through  $\hat{V}_{SB} = \sum_\alpha \hat{a}^\dagger \hat{B}_\alpha + \hat{a} \hat{B}_\alpha^\dagger$ . Here  $\hat{B}_\alpha = \sum g_{\alpha k} \hat{b}_{\alpha k}$  is the operator of bath- $\alpha$ , and the initial states of the baths are squeezed thermal baths (hereafter, all the density matrices are written in the interaction picture),

$$\begin{aligned} \rho_{B\alpha}^0 &= \frac{1}{\mathcal{Z}_\alpha} e^{-\beta_\alpha \mathcal{S}_\alpha \hat{H}_{B\alpha} \mathcal{S}_\alpha^\dagger}, \quad \beta_\alpha := T_\alpha^{-1}, \\ \mathcal{S}_\alpha &:= \prod_k \exp \left[ \frac{1}{2} \lambda_{\alpha k}^* \hat{b}_{\alpha k}^2 - \text{H.c.} \right], \quad \lambda_{\alpha k} = r_{\alpha k} e^{-i\theta_{\alpha k}}. \end{aligned} \quad (7)$$

Here  $\mathcal{S}_\alpha$  is the squeezing operator for the boson modes in bath- $\alpha$ . With the Born-Markovian approximation, we obtain a master equation  $\dot{\rho} = \sum_\alpha \mathcal{L}_\alpha[\rho]$  for the open system alone [26,29], where

$$\begin{aligned} \mathcal{L}_\alpha[\rho] &= \frac{\gamma_\alpha}{2} [\tilde{n}_\alpha (2\hat{a}^\dagger \rho \hat{a} - \{\hat{a} \hat{a}^\dagger, \rho\}) + (\tilde{n}_\alpha + 1) (2\hat{a} \rho \hat{a}^\dagger \\ &\quad - \{\hat{a}^\dagger \hat{a}, \rho\}) - \tilde{u}_\alpha (2\hat{a}^\dagger \rho \hat{a}^\dagger - \{(\hat{a}^\dagger)^2, \rho\}) \\ &\quad - \tilde{u}_\alpha^* (2\hat{a} \rho \hat{a} - \{\hat{a}^2, \rho\})]. \end{aligned}$$

The coupling spectra of the squeezed bath- $\alpha$  are  $J_\alpha(\omega) := 2\pi \sum_k |g_{\alpha k}|^2 \delta(\omega - \omega_{\alpha k})$  and  $K_\alpha(\omega) := 2\pi \sum_k g_{\alpha k}^2 \delta(\omega - \omega_{\alpha k})$ . Without loss of generality, we omit the phase of  $g_{\alpha k}$  and thus  $K_\alpha(\omega) = K_\alpha^*(\omega) = J_\alpha(\omega)$ . Here we denote  $\gamma_\alpha := J_\alpha(\Omega) = K_\alpha(\Omega)$ , and the parameters  $\tilde{n}_\alpha := \tilde{n}_\alpha(\Omega)$ ,  $\tilde{u}_\alpha := \tilde{u}_\alpha(\Omega)$  are calculated from  $\tilde{n}_\alpha(\omega_k) := \text{tr}[\rho_{B\alpha}^0 \hat{b}_{\alpha k}^\dagger \hat{b}_{\alpha k}]$ ,  $\tilde{u}_\alpha(\omega_k) := -\text{tr}[\rho_{B\alpha}^0 \hat{b}_{\alpha k}^2]$  (see Appendix B). The master equation gives

$$\begin{aligned} \frac{d}{dt} \langle \tilde{a}(t) \rangle &= -\sum_\alpha \frac{\gamma_\alpha}{2} \langle \tilde{a} \rangle, \quad \frac{d}{dt} \langle \tilde{a}^2 \rangle = -\sum_\alpha \gamma_\alpha [\langle \tilde{a}^2 \rangle - \tilde{u}_\alpha], \\ \frac{d}{dt} \langle \tilde{a}^\dagger \tilde{a} \rangle &= -\sum_\alpha \gamma_\alpha [\langle \tilde{n}_\alpha \rangle - \tilde{n}_\alpha]. \end{aligned} \quad (8)$$

Here we denote  $\hat{n}_\alpha := \hat{a}^\dagger \hat{a}$ , and  $\langle \tilde{o}(t) \rangle := \text{tr}[\rho \hat{o}(t)]$  gives variables in the rotating frame.<sup>2</sup>

<sup>2</sup>Here  $\rho$  is in the interaction picture, but  $\hat{o}$  is in the Schrödinger picture, thus we have  $\langle \hat{o}(t) \rangle = \langle \tilde{o}(t) \rangle e^{-i\Omega t}$ . Here  $\langle \tilde{o}(t) \rangle$  stands for observable expectations, which are independent of pictures, and  $\langle \hat{o}(t) \rangle$  are variables in the rotating frame, thus in Eq. (8) the dependence of the system frequency  $\Omega$  is canceled.

The partial steady states  $\rho_{ss}^{(\alpha)}$ , which satisfy  $\mathcal{L}_\alpha[\rho_{ss}^{(\alpha)}] = 0$ , are now squeezed thermal states,

$$\begin{aligned} \rho_{ss}^{(\alpha)} &= \frac{1}{Z_\alpha} \exp[-\beta_\alpha \Omega \mathcal{S}_\alpha \hat{a}^\dagger \hat{a} \mathcal{S}_\alpha^\dagger], \\ \mathcal{S}_\alpha &:= \exp \left[ - \left( \frac{1}{2} \zeta_\alpha^* \hat{a}^2 - \text{H.c.} \right) \right], \quad \zeta_\alpha = \lambda_{\alpha k} \Big|_{\omega_k = \Omega} := r_\alpha e^{i\theta_\alpha}. \end{aligned} \quad (9)$$

Here  $\mathcal{S}_\alpha$  is a squeezing operator for the system. Although the baths are not thermal, we can still write down the Spohn formula  $R_{Sp} = \sum_\alpha R_{Sp}^{(\alpha)}$ , where

$$\begin{aligned} R_{Sp}^{(\alpha)} &:= \text{tr}[\mathcal{L}_\alpha[\rho](\ln \rho_{ss}^{(\alpha)} - \ln \rho)] \\ &:= \chi_\alpha - \text{tr}[\mathcal{L}_\alpha[\rho] \ln \rho], \end{aligned} \quad (10)$$

and we can prove that  $R_{Sp}^{(\alpha)} \geq 0$  and  $R_{Sp} \geq 0$  also hold in this nonthermal case (Appendix A).

However, since the above Spohn formula  $R_{Sp}$  for nonthermal baths no longer comes from the thermodynamic EPr (1), its physical meaning is unclear now. In the thermal case, the first term in  $R_{Sp}^{(\alpha)}$ ,  $\chi_\alpha := \text{tr}[\mathcal{L}_\alpha[\rho] \ln \rho_{ss}^{(\alpha)}]$ , gives the changing rate of the thermal entropy ( $\chi_\alpha = -\dot{Q}_\alpha/T_\alpha$ ). But for the squeezed case, it becomes

$$\begin{aligned} \chi_\alpha &= \frac{\Omega}{T_\alpha} \gamma_\alpha (\cosh 2r_\alpha [\langle \tilde{n}_\alpha(t) \rangle - \tilde{n}_\alpha] - \frac{1}{2} \sinh 2r_\alpha \\ &\quad \times \{e^{-i\theta_\alpha} [\langle \tilde{a}^2(t) \rangle - \tilde{u}_\alpha] + \text{H.c.}\}). \end{aligned} \quad (11)$$

It is difficult to tell the physical meaning of this quantity. In the following, we will show that indeed Eq. (11) is just the changing rate of the von Neumann entropy of bath- $\alpha$ , i.e.,  $\chi_\alpha = \dot{S}_{B\alpha}$ , and then Eq. (10) leads directly to

$$R_{Sp} = \sum_\alpha R_{Sp}^{(\alpha)} = \dot{S}_S + \sum_\alpha \dot{S}_{B\alpha} = R_{\mathcal{I}}. \quad (12)$$

## B. Bath entropy dynamics

Now we are going to calculate the entropy changing rate  $\dot{S}_{B\alpha}$  of bath- $\alpha$  directly. To do this, we adopt the same trick as in the thermal case. Assuming the squeezed baths do not change too much (interaction picture), the entropy of the bath evolves as

$$\begin{aligned} \frac{d}{dt} S[\rho_{B\alpha}(t)] &\simeq -\text{tr} \left[ \dot{\rho}_{B\alpha}(t) \ln \frac{\exp[-\beta_\alpha \mathcal{S}_\alpha \hat{H}_{B\alpha} \mathcal{S}_\alpha^\dagger]}{Z_\alpha} \right] \\ &= \frac{d}{dt} \sum_k \frac{\omega_{\alpha k}}{T_\alpha} (\cosh 2r_{\alpha k} \langle \tilde{b}_{\alpha k}^\dagger(t) \tilde{b}_{\alpha k}(t) \rangle \\ &\quad + \frac{1}{2} \sinh 2r_{\alpha k} [\langle \tilde{b}_{\alpha k}^2(t) \rangle e^{-i\theta_{\alpha k}} + \text{H.c.}]). \end{aligned} \quad (13)$$

Thus, the calculation of the bath entropy is now reduced to calculating the time derivative of the expectations of the bath operators such as  $\langle \tilde{b}_{\alpha k}^\dagger(t) \tilde{b}_{\alpha k}(t) \rangle$  and  $\langle \tilde{b}_{\alpha k}^2(t) \rangle$ .

This can be done with the help of the Heisenberg equations,  $\dot{\hat{b}}_{\alpha k} = -i\omega_{\alpha k} \hat{b}_{\alpha k} - i g_{\alpha k}^* \hat{a}$ , and  $\dot{\hat{a}} = -i\Omega \hat{a} -$

$i \sum_\alpha g_{\alpha k} \hat{b}_{\alpha k}$ , which lead to the quantum Langevin equation [29–31]

$$\frac{d}{dt} \hat{a} = -i\Omega \hat{a} - \frac{1}{2} \Gamma \hat{a} - \hat{\mathcal{E}}(t). \quad (14)$$

Here  $\Gamma := \sum_\alpha \gamma_\alpha$  is the total decay rate, and  $\gamma_\alpha$  are the same as those in the master equation;  $\hat{\mathcal{E}}(t) := \sum_\alpha \hat{\xi}_\alpha(t)$  is the random force, and  $\hat{\xi}_\alpha(t) := i \sum_k g_{\alpha k} \hat{b}_{\alpha k}(0) e^{-i\omega_{\alpha k} t}$  is the contribution from bath- $\alpha$ . Thus  $\hat{a}(t)$  and  $\hat{b}_{\alpha k}(t)$  evolve as

$$\begin{aligned} \hat{a}(t) &= \hat{a}(0) e^{-i\Omega t - \frac{\Gamma}{2} t} - \int_0^t ds e^{-i\Omega(t-s) - \frac{\Gamma}{2}(t-s)} \hat{\mathcal{E}}(s), \\ \hat{b}_{\alpha k}(t) &= \hat{b}_{\alpha k}(0) e^{-i\omega_{\alpha k} t} - i g_{\alpha k}^* \int_0^t ds e^{-i\omega_{\alpha k}(t-s)} \hat{a}(s). \end{aligned} \quad (15)$$

To further calculate the bath entropy change, we are now going to demonstrate that the following two relations hold in the weak-coupling limit and the Markovian approximation:

$$\begin{aligned} \frac{d}{dt} \sum_k \mathfrak{f}_k \langle \tilde{b}_{\alpha k}^\dagger \tilde{b}_{\alpha k} \rangle &\simeq \mathfrak{f}(\Omega) \gamma_\alpha [\langle \tilde{n}_\alpha \rangle - \tilde{n}_\alpha], \\ \frac{d}{dt} \sum_k \mathfrak{h}_k \langle \tilde{b}_{\alpha k}^2 \rangle + \text{H.c.} &\simeq -\mathfrak{h}(\Omega) \gamma_\alpha [\langle \tilde{a}^2 \rangle - \tilde{u}_\alpha] + \text{H.c.}, \end{aligned} \quad (16)$$

where  $\mathfrak{f}_k$  and  $\mathfrak{h}_k$  are arbitrary coefficients depending on  $k$ .

If we set  $\mathfrak{f}_k = \frac{\omega_{\alpha k}}{T_\alpha} \cosh 2r_{\alpha k}$ ,  $\mathfrak{h}_k = \frac{\omega_{\alpha k}}{2T_\alpha} \sinh 2r_{\alpha k} e^{-i\theta_{\alpha k}}$ , and sum up the above two equations, then the left side simply gives  $\dot{S}_{B\alpha}$  [Eq. (13)]; at the same time, the right side is just equal to  $\chi_\alpha$  [Eq. (11)]. Thus we can prove  $\chi_\alpha = \dot{S}_{B\alpha}$ , i.e., the term  $\chi_\alpha = \text{tr}[\ln \rho_{ss}^{(\alpha)} \mathcal{L}_\alpha[\rho]]$  in the Spohn formula is just the changing rate of the von Neumann entropy of bath- $\alpha$ .

In addition, if we set  $\mathfrak{f}_k = \omega_{\alpha k}$  and  $\mathfrak{h}_k = 0$ , the above relations lead to  $\frac{d}{dt} \langle \hat{H}_{B\alpha} \rangle = \Omega \gamma_\alpha [\langle \tilde{n}_\alpha \rangle - \tilde{n}_\alpha] = -\dot{Q}_\alpha$ , which means the energy loss of bath- $\alpha$  is equal to the energy gain of the system from bath- $\alpha$  [as we utilized in the discussion below Eq. (5)].

The calculation of Eq. (16) goes as follows:

$$\begin{aligned} \frac{d}{dt} \sum_k \mathfrak{f}_k \langle \tilde{b}_{\alpha k}^\dagger \tilde{b}_{\alpha k} \rangle &= \sum_k \mathfrak{f}_k i g_{\alpha k} \langle \dot{\hat{a}}^\dagger \hat{b}_{\alpha k} \rangle + \text{H.c.} \\ &= \sum_k \mathfrak{f}_k \left[ i g_{\alpha k} \langle \dot{\hat{a}}^\dagger(t) \hat{b}_{\alpha k}(0) \rangle e^{-i\omega_{\alpha k} t} + |g_{\alpha k}|^2 \right. \\ &\quad \left. \times \int_0^t ds e^{-i\omega_{\alpha k}(t-s)} \langle \hat{a}^\dagger(t) \hat{a}(s) \rangle \right] + \text{H.c.} \end{aligned} \quad (17)$$

The first term in the brackets can be further calculated by substituting  $\hat{a}(t)$  [Eq. (15)],

$$\begin{aligned} &\sum_k \mathfrak{f}_k i g_{\alpha k} \langle \dot{\hat{a}}^\dagger(t) \hat{b}_{\alpha k}(0) \rangle e^{-i\omega_{\alpha k} t} + \text{H.c.} \\ &= - \sum_k \mathfrak{f}_k |g_{\alpha k}|^2 \int_0^t ds e^{[i(\Omega - \omega_k) - \frac{\Gamma}{2}](t-s)} \langle \hat{b}_{\alpha k}^\dagger(0) \hat{b}_{\alpha k}(0) \rangle + \text{H.c.} \\ &= - \int_0^t ds \left[ \int_0^\infty \frac{d\omega}{2\pi} e^{[i(\Omega - \omega) - \frac{\Gamma}{2}](t-s)} J_\alpha(\omega) \mathfrak{f}(\omega) \tilde{n}_\alpha(\omega) \right] + \text{H.c.} \end{aligned} \quad (18)$$

Assuming the frequency integral in the brackets gives a fast-decaying function of  $(t - s)$ , we extend the time integral to  $t \rightarrow \infty$  (Markovian approximation), and that gives

$$\begin{aligned} & - \int_0^\infty \frac{d\omega}{2\pi} \left[ \int_0^\infty ds e^{i(\Omega-\omega)s - \frac{1}{2}\Gamma s} \right] J_\alpha(\omega) \mathfrak{f}(\omega) \tilde{n}_\alpha(\omega) + \text{H.c.} \\ & = - \int_0^\infty \frac{d\omega}{2\pi} J_\alpha(\omega) \mathfrak{f}(\omega) \tilde{n}_\alpha(\omega) \frac{\Gamma}{\left(\frac{\Gamma}{2}\right)^2 + (\omega - \Omega)^2} \\ & \simeq -\mathfrak{f}(\Omega) \gamma_\alpha \tilde{n}_\alpha. \end{aligned} \quad (19)$$

The last line holds in the weak-coupling limit  $\Gamma \ll \Omega$  because the Lorentzian function in the integral approaches  $2\pi\delta(\omega - \Omega)$ .

To calculate the second term of Eq. (17), we should notice that  $\langle \hat{a}^\dagger(t) \hat{a}(s) \rangle = \langle \tilde{a}^\dagger(s) \tilde{a}(s) \rangle e^{i(\Omega - \frac{\Gamma}{2})(t-s)}$  holds for  $t \geq s$  (quantum regression theorem [26,30]). Here  $\langle \tilde{\delta}_1(t) \tilde{\delta}_2(s) \rangle$  is a correlation function in the rotating frame, defined by  $\langle \tilde{\delta}_1(t) \tilde{\delta}_2(s) \rangle = \text{tr}[\hat{\delta}_1 \mathcal{E}_{t-s} \hat{\delta}_2 \mathcal{E}_s \rho(0)]$  for  $t \geq s$  [26], where  $\hat{\delta}_{1,2}$  are operators in the Schrödinger picture,  $\mathcal{E}_t$  is the evolution operator solved from the above master equation in the interaction picture, and  $\rho(t) = \mathcal{E}_{t-s} \rho(s)$ . Similarly,  $\langle \hat{\delta}_1(t) \hat{\delta}_2(s) \rangle$  are correlation functions in the nonrotating frame. Thus the second term of Eq. (17) gives

$$\begin{aligned} & \sum_k \mathfrak{f}_k |g_{\alpha k}|^2 \int_0^t ds e^{-i\omega_\alpha(t-s)} \langle \hat{a}^\dagger(t) \hat{a}(s) \rangle + \text{H.c.} \\ & \simeq \int_0^\infty \frac{d\omega}{2\pi} \mathfrak{f}(\omega) J_\alpha(\omega) \langle \tilde{n}_\alpha(t) \rangle \int_0^\infty ds e^{i(\Omega-\omega)s - \frac{\Gamma}{2}s} + \text{H.c.} \\ & = \langle \tilde{n}_\alpha(t) \rangle \int_0^\infty \frac{d\omega}{2\pi} \mathfrak{f}(\omega) J_\alpha(\omega) \frac{\Gamma}{\left(\frac{\Gamma}{2}\right)^2 + (\omega - \Omega)^2} \\ & \simeq \gamma_\alpha \mathfrak{f}(\Omega) \langle \tilde{n}_\alpha(t) \rangle. \end{aligned} \quad (20)$$

Again we adopted the Markovian approximations as before, and  $\langle \tilde{n}_\alpha(s) \rangle$  is taken out of the integral directly.

Therefore, summing up Eqs. (19) and (20), we obtain the first relation in Eq. (16). The second relation can be obtained in a similar way (see Appendix B). Then, by setting proper coefficients  $\mathfrak{f}_k$  and  $\mathfrak{h}_k$  in Eq. (16), we can prove  $\chi_\alpha = \hat{S}_{B\alpha}$ , and further  $R_{\mathcal{I}} = R_{\text{Sp}}$ . Since we can prove the Spohn formula  $R_{\text{Sp}} \geq 0$ , the MIPr  $R_{\mathcal{I}}$  also remains positive, which means that the system-bath mutual information, or their correlation, continues to increase monotonically in this nonthermal case.

#### IV. SUMMARY

In this paper, we study the production of mutual information between a system and its environment. We find that this MIPr (4) has a close connection with the conventional thermodynamic description of the EPr (1): when the baths of the open system are canonical thermal baths, this MIPr could return exactly to the previous EPr. Therefore, the second law statement  $R_{\text{ep}} \geq 0$  can be understood equivalently as saying the system-bath correlation always keeps increasing.

In addition, we also study an example of a single boson mode coming into contact with multiple squeezed thermal baths. In this case, the temperatures of the baths are not well defined, and the previous EPr does not apply. We proved that the MIPr is still positive, which means the monotonic increase

of the system-bath correlation also exists in this case. It is definitely worthwhile to study the MIPr in more nonthermal systems.

We remark that the proof for the positivity of the MIPr and the Spohn formula relies on the fact that the dynamics of the system can be well described by a Markovian master equation with the Lindblad (GKSL) form. If this is not fulfilled [22,28,32,33], the positivity cannot be guaranteed.

Our study indicates that it is the system-bath correlation that keeps increasing monotonically, although the total  $S + B$  system evolves unitarily. This idea is also consistent with some other fundamental studies on thermodynamics, such as the local relaxation hypothesis [34], entanglement-based thermodynamics [35], and the mutual information understanding of Blackhole radiation [36].

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#### APPENDIX A: PROOF FOR THE POSITIVITY OF THE SPOHN FORMULA

Now we prove that the Spohn formula  $R_{\text{Sp}}$  is positive also for nonthermal baths. Namely, for a Lindblad (GKSL) master equation such as [5,6]

$$\dot{\rho} = i[\rho, \hat{H}_S] + \sum_\alpha \mathcal{L}_\alpha[\rho] := i[\rho, \hat{H}_S] + \mathcal{L}[\rho],$$

$$\mathcal{L}_\alpha[\rho] = \sum_n V_{\alpha,n} \rho V_{\alpha,n}^\dagger - \frac{1}{2} \{V_{\alpha,n}^\dagger V_{\alpha,n}, \rho\}, \quad (\text{A1})$$

we have

$$R_{\text{Sp}}^{(\alpha)} = \text{tr}[\mathcal{L}_\alpha[\rho] (\ln \rho_{\text{ss}}^{(\alpha)} - \ln \rho)] \geq 0,$$

$$R_{\text{Sp}} = \sum_\alpha R_{\text{Sp}}^{(\alpha)} \geq 0, \quad (\text{A2})$$

where  $\rho_{\text{ss}}^{(\alpha)}$  is the partial steady state satisfying  $\mathcal{L}_\alpha[\rho_{\text{ss}}^{(\alpha)}] = 0$ . The operator  $\mathcal{L}_\alpha[\rho]$  describes the dissipation to bath- $\alpha$ , which does not have to be a thermal bath, and  $\rho_{\text{ss}}^{(\alpha)}$  is not necessarily a thermal state.

Our proof follows from Ref. [7], where a single bath was concerned and the EPr was defined by the relative entropy [15,37],

$$\sigma = -\frac{d}{dt} S[\rho(t) \parallel \rho_{\text{ss}}] = \text{tr}[\mathcal{L}[\rho] (\ln \rho_{\text{ss}} - \ln \rho)]. \quad (\text{A3})$$

Here  $\rho_{\text{ss}}$  is the steady state of the system satisfying  $\mathcal{L}[\rho_{\text{ss}}] = 0$ . This is equivalent to Eq. (A2) when only one single bath is concerned. This EPr based on relative entropy always gives  $\sigma = 0$  at the steady state, even for the nonequilibrium steady state when there are multiple baths, and usually a steady nonequilibrium flux exists. However, the EPr we

used [Eq. (A2)] will remain nonzero in this case, which means the irreversible entropy is still being produced in the nonequilibrium steady state.

The proof for the positivity of Eq. (A2) is as follows.

*Proof:* Since the master equation has Lindblad (GKSL) form, we obtain

$$\mathrm{tr}[\mathcal{L}_\alpha[\rho] \ln \rho] = \sum_n \mathrm{tr}[V_{\alpha,n} \rho V_{\alpha,n}^\dagger \ln \rho - V_{\alpha,n}^\dagger V_{\alpha,n} \rho \ln \rho]. \quad (\text{A4})$$

Now we need the Lieb theorem [38], i.e., the functional  $f_q^{(V)}[\rho] = -\mathrm{tr}[\rho^q V \rho^{1-q} V^\dagger]$  is convex for  $\forall 0 \leq q \leq 1$ , i.e.,

$$f_q^{(V)}[\lambda_1 \rho_1 + \lambda_2 \rho_2] \leq \lambda_1 f_q^{(V)}[\rho_1] + \lambda_2 f_q^{(V)}[\rho_2]. \quad (\text{A5})$$

At  $q = 0$ ,  $f_{q=0}[\rho] = -\mathrm{tr}[V \rho V^\dagger]$  is a linear map satisfying  $f_0[\lambda_1 \rho_1 + \lambda_2 \rho_2] = \lambda_1 f_0[\rho_1] + \lambda_2 f_0[\rho_2]$ , therefore the derivative  $\partial_q f_q^{(V)} := \epsilon^{-1}[f_{q+\epsilon}^{(V)} - f_q^{(V)}]$  is also convex around  $q = 0$ , which reads

$$\partial_q f_q^{(V)}[\rho]|_{q=0} = \mathrm{tr}[\rho^q V \rho^{1-q} \ln \rho V^\dagger - \rho^q \ln \rho V \rho^{1-q} V^\dagger]|_{q=0} = \mathrm{tr}[V^\dagger V \rho \ln \rho - V \rho V^\dagger \ln \rho] := -\mathrm{tr}[\hat{\mathbf{L}}_V[\rho] \cdot \ln \rho]. \quad (\text{A6})$$

Here we denoted  $\hat{\mathbf{L}}_V[\rho] := V \rho V^\dagger - \frac{1}{2}\{V^\dagger V, \rho\}$ . Thus, we obtain the following relation ( $\lambda \geq 0$ ):

$$\begin{aligned} \partial_q f_q^{(V)}[\lambda \rho + (1-\lambda)\rho_{\mathrm{ss}}^{(\alpha)}] &= -\mathrm{tr}[\hat{\mathbf{L}}_V[\lambda \rho + (1-\lambda)\rho_{\mathrm{ss}}^{(\alpha)}] \ln[\lambda \rho + (1-\lambda)\rho_{\mathrm{ss}}^{(\alpha)}]] \\ &\leq \lambda \partial_q f_q^{(V)}[\rho] + (1-\lambda) \partial_q f_q^{(V)}[\rho_{\mathrm{ss}}^{(\alpha)}] = -\lambda \mathrm{tr}[\hat{\mathbf{L}}_V[\rho] \ln \rho] - (1-\lambda) \mathrm{tr}[\hat{\mathbf{L}}_V[\rho_{\mathrm{ss}}^{(\alpha)}] \ln \rho_{\mathrm{ss}}^{(\alpha)}]. \end{aligned} \quad (\text{A7})$$

Since the Lindblad operator can be written as  $\mathcal{L}_\alpha[\rho] = \sum_n \hat{\mathbf{L}}_{V_{\alpha,n}}[\rho]$ , from the above relation we obtain

$$-\mathrm{tr}[\mathcal{L}_\alpha[\lambda \rho + (1-\lambda)\rho_{\mathrm{ss}}^{(\alpha)}] \ln[\lambda \rho + (1-\lambda)\rho_{\mathrm{ss}}^{(\alpha)}]] \leq -\lambda \mathrm{tr}[\mathcal{L}_\alpha[\rho] \ln \rho] - (1-\lambda) \mathrm{tr}[\mathcal{L}_\alpha[\rho_{\mathrm{ss}}^{(\alpha)}] \ln \rho_{\mathrm{ss}}^{(\alpha)}]. \quad (\text{A8})$$

Here  $\mathcal{L}_\alpha$  is a linear operator, thus  $\mathcal{L}_\alpha[\lambda \rho + (1-\lambda)\rho_{\mathrm{ss}}^{(\alpha)}] = \lambda \mathcal{L}_\alpha[\rho] + (1-\lambda) \mathcal{L}_\alpha[\rho_{\mathrm{ss}}^{(\alpha)}]$ . Remember that we require  $\mathcal{L}_\alpha[\rho_{\mathrm{ss}}^{(\alpha)}] = 0$ , thus the above inequality becomes

$$-\lambda \mathrm{tr}[\mathcal{L}_\alpha[\rho] \ln[\lambda \rho + (1-\lambda)\rho_{\mathrm{ss}}^{(\alpha)}]] \leq -\lambda \mathrm{tr}[\mathcal{L}_\alpha[\rho] \ln \rho]. \quad (\text{A9})$$

In the limit  $\lambda \rightarrow 0^+$ , we obtain

$$\mathrm{tr}[\mathcal{L}_\alpha[\rho] (\ln \rho_{\mathrm{ss}}^{(\alpha)} - \ln \rho)] = R_{\mathrm{Sp}}^{(\alpha)} \geq 0. \quad (\text{A10})$$

Therefore, we have  $R_{\mathrm{Sp}} = \sum R_{\mathrm{Sp}}^{(\alpha)} \geq 0$ . ■

## APPENDIX B: PROPERTIES OF A SQUEEZED BATH AND THE MASTER EQUATION

### 1. Squeezed bath properties

Here we show some basic properties of a squeezed thermal bath. The Hamiltonian of the bath is  $\hat{H}_B = \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k$ , and the squeezed thermal state is

$$\rho_B := \mathcal{S} \rho_{\mathrm{th}} \mathcal{S}^\dagger, \quad \rho_{\mathrm{th}} := \frac{1}{\mathcal{Z}} \exp[-\beta \hat{H}_B]. \quad (\text{B1})$$

Here  $\mathcal{S}$  is the squeezing operator for the boson bath,

$$\mathcal{S} := \prod_k \mathfrak{s}_k(\lambda_k), \quad \mathfrak{s}_k(\lambda_k) := \exp\left[\frac{1}{2} \lambda_k^* \hat{b}_k^2 - \text{H.c.}\right], \quad \lambda_k := r_k e^{i\theta_k} (r_k > 0), \quad (\text{B2})$$

and  $\mathfrak{s}_k$  is the squeezing operator for mode  $\hat{b}_k$  in the bath. They satisfy

$$\begin{aligned} \mathfrak{s}_k^\dagger(\lambda_k) \hat{b}_k \mathfrak{s}_k(\lambda_k) &= \hat{b}_k + \left[\frac{1}{2}[\lambda_k(\hat{b}_k^\dagger)^2 - \lambda_k^* \hat{b}_k^2], \hat{b}_k\right] + \frac{1}{2!} \left[\frac{1}{2}[\lambda_k(\hat{b}_k^\dagger)^2 - \lambda_k^* \hat{b}_k^2], \left[\frac{1}{2}[\lambda_k(\hat{b}_k^\dagger)^2 - \lambda_k^* \hat{b}_k^2], \hat{b}_k\right]\right] + \dots \\ &= \hat{b}_k - \lambda_k \hat{b}_k^\dagger + \frac{1}{2!} |\lambda_k|^2 \hat{b}_k - \frac{1}{3!} \lambda_k |\lambda_k|^2 \hat{b}_k^\dagger - \frac{1}{4!} |\lambda_k|^4 \hat{b}_k + \dots = \hat{b}_k \cosh r_k - \hat{b}_k^\dagger e^{i\theta_k} \sinh r_k, \\ \mathfrak{s}_k(\lambda_k) \hat{b}_k \mathfrak{s}_k^\dagger(\lambda_k) &= \hat{b}_k \cosh r_k + \hat{b}_k^\dagger e^{i\theta_k} \sinh r_k. \end{aligned} \quad (\text{B3})$$

Thus we have

$$\begin{aligned}\tilde{u}_k &:= -\text{tr}[\rho_B \hat{b}_k^2] = -\text{tr}[\rho_{\text{th}} \hat{s}_k^\dagger \hat{b}_k \hat{s}_k^\dagger \hat{b}_k \hat{s}_k] = -\text{tr}[\rho_{\text{th}} (\hat{b}_k \cosh r_k - \hat{b}_k^\dagger e^{i\theta_k} \sinh r_k) (\hat{b}_k \cosh r_k - \hat{b}_k^\dagger e^{i\theta_k} \sinh r_k)] \\ &= \cosh r_k \sinh r_k e^{i\theta_k} (2\bar{n}_k + 1) = e^{i\theta_k} \sinh 2r_k \left( \bar{n}_k + \frac{1}{2} \right),\end{aligned}\quad (\text{B4})$$

$$\begin{aligned}\tilde{n}_k &:= \text{tr}[\rho_B \hat{b}_k^\dagger \hat{b}_k] = \text{tr}[\rho_{\text{th}} \hat{s}_k^\dagger \hat{b}_k^\dagger \hat{s}_k \hat{s}_k^\dagger \hat{b}_k \hat{s}_k] = \text{tr}[\rho_{\text{th}} (\hat{b}_k^\dagger \cosh r_k - \hat{b}_k e^{-i\theta_k} \sinh r_k) (\hat{b}_k \cosh r_k - \hat{b}_k^\dagger e^{i\theta_k} \sinh r_k)] \\ &= \cosh^2 r_k \bar{n}_k + \sinh^2 r_k (\bar{n}_k + 1) = \cosh 2r_k \left( \bar{n}_k + \frac{1}{2} \right) - \frac{1}{2},\end{aligned}\quad (\text{B5})$$

where  $\bar{n}_k := [\exp(\beta\omega_k) - 1]^{-1}$  is the Planck distribution.

## 2. Master equation derivation

Now we derive the master equation for a single boson mode ( $\hat{H}_S = \Omega \hat{a}^\dagger \hat{a}$ ) interacting with a squeezed boson bath. The interaction Hamiltonian is  $\hat{V}_{SB} = \hat{a} \hat{B}^\dagger + \hat{a}^\dagger \hat{B}$ , where  $\hat{B} = \sum_k g_k \hat{b}_k$ , and the master equation is derived by

$$\begin{aligned}\dot{\rho} &= -\text{tr}_B \int_0^\infty ds [\tilde{V}_{SB}(t-s), [\tilde{V}_{SB}(t), \rho(t) \otimes \rho_B]] \\ &= \text{tr}_B \int_0^\infty ds [\tilde{V}_{SB}(t-s) \rho(t) \rho_B \tilde{V}_{SB}(t) - \tilde{V}_{SB}(t-s) \tilde{V}_{SB}(t) \rho(t) \rho_B] + \text{H.c.}\end{aligned}\quad (\text{B6})$$

Here we use  $\tilde{o}(t)$  to denote the operators in the interaction picture, and  $\tilde{a}(t) = \hat{a} e^{-i\Omega t}$ ,  $\tilde{b}_k(t) = \hat{b}_k e^{-i\omega_k t}$ . We adopted the Born approximation  $\rho_{SB}(t) \simeq \rho(t) \otimes \rho_B$ , and

$$\rho_B(t) \simeq \rho_B^0 = \frac{1}{Z} \exp[-\beta \mathcal{S} \hat{H}_B \mathcal{S}^\dagger].\quad (\text{B7})$$

We define the coupling spectrum as

$$J(\omega) := \sum_k |g_k|^2 \delta(\omega - \omega_k), \quad K(\omega) := \sum_k g_k^2 \delta(\omega - \omega_k).\quad (\text{B8})$$

We omit the phase of  $g_k$ , thus we have  $K(\omega) = J(\omega) = K^*(\omega)$ . Here is the calculation for some terms:

$$\begin{aligned}\int_0^\infty ds \text{tr}_B [\tilde{a}^\dagger(t-s) \tilde{B}(t-s) \rho(t) \rho_B \tilde{a}(t) \tilde{B}^\dagger(t)] &= \int_0^\infty ds \hat{a}^\dagger \rho \hat{a} e^{-i\Omega s} \text{tr}_B [\rho_B \tilde{B}^\dagger(t) \tilde{B}(t-s)] \\ &= \hat{a}^\dagger \rho \hat{a} \int_0^\infty \frac{d\omega}{2\pi} \int_0^\infty ds e^{-i\Omega s} e^{i\omega s} J(\omega) \bar{n}(\omega) = \frac{1}{2} \gamma \bar{n} \hat{a}^\dagger \rho \hat{a},\end{aligned}\quad (\text{B9})$$

$$\begin{aligned}\int_0^\infty ds \text{tr}_B [\tilde{a}^\dagger(t-s) \tilde{B}(t-s) \rho(t) \rho_B \tilde{a}^\dagger(t) \tilde{B}(t)] &= \int_0^\infty ds \hat{a}^\dagger \rho \hat{a}^\dagger e^{2i\Omega t} e^{-i\Omega s} \text{tr}_B [\rho_B \tilde{B}(t) \tilde{B}(t-s)] \\ &= -\hat{a}^\dagger \rho \hat{a}^\dagger e^{2i\Omega t} \int_0^\infty \frac{d\omega}{2\pi} \int_0^\infty ds e^{-i\Omega s} e^{i\omega s} e^{-2i\omega t} K(\omega) \tilde{u}(\omega) = -\frac{1}{2} \gamma \tilde{u} \hat{a}^\dagger \rho \hat{a}^\dagger,\end{aligned}\quad (\text{B10})$$

where  $\gamma = J(\Omega) = K(\Omega)$  and

$$\bar{n} = \cosh 2r_\Omega \left( \bar{n}_\Omega + \frac{1}{2} \right) - \frac{1}{2}, \quad \tilde{u} = e^{i\theta_\Omega} \sinh 2r_\Omega \left( \bar{n}_\Omega + \frac{1}{2} \right).\quad (\text{B11})$$

We omitted all the principal integrals in the above calculation. Thus the master equation is (interaction picture)

$$\begin{aligned}\dot{\rho} &= \gamma \bar{n} \left( \hat{a}^\dagger \rho \hat{a} - \frac{1}{2} \{ \hat{a} \hat{a}^\dagger, \rho \} \right) + \gamma (\bar{n} + 1) \left( \hat{a} \rho \hat{a}^\dagger - \frac{1}{2} \{ \hat{a}^\dagger \hat{a}, \rho \} \right) \\ &\quad - \gamma \tilde{u} \left( \hat{a}^\dagger \rho \hat{a}^\dagger - \frac{1}{2} \{ (\hat{a}^\dagger)^2, \rho \} \right) + \gamma \tilde{u}^* \left( \hat{a} \rho \hat{a} - \frac{1}{2} \{ (\hat{a})^2, \rho \} \right).\end{aligned}\quad (\text{B12})$$

From the above master equation, we obtain

$$\frac{d}{dt} \langle \tilde{a}(t) \rangle = -\frac{\gamma}{2} \langle \tilde{a} \rangle, \quad \frac{d}{dt} \langle \tilde{a}^\dagger \tilde{a} \rangle = -\gamma [\langle \tilde{n}_a \rangle - \bar{n}], \quad \frac{d}{dt} \langle \tilde{a}^2 \rangle = -\gamma [\langle \tilde{a}^2 \rangle - \tilde{u}].\quad (\text{B13})$$

In the steady state, we have  $\langle \tilde{a} \rangle_{ss} = 0$ ,  $\langle \tilde{a}^\dagger \tilde{a} \rangle_{ss} = \tilde{n}$ , and  $\langle \tilde{a}^2 \rangle_{ss} = \tilde{u}$ . Thus we can verify that the steady state is

$$\rho_{ss} = \frac{1}{Z} \exp[-\beta \Omega \mathbf{S} \hat{a}^\dagger \hat{a} \mathbf{S}^\dagger], \quad \mathbf{S} = \exp \left[ -\frac{1}{2} \zeta^* \hat{a}^2 + \frac{1}{2} \zeta (\hat{a}^\dagger)^2 \right], \quad \zeta := \lambda_k \Big|_{\omega_k = \Omega}. \quad (\text{B14})$$

Here  $\mathbf{S}$  is a squeezing operator for the system, and we remark that the above  $\rho_{ss}$  is in the interaction picture. When the single boson is coupled with multiple squeezed baths, the generalization is straightforward, as shown in the main text.

### 3. Time correlation functions

From the above equations of  $\langle \tilde{a}(t) \rangle$ , we obtain  $\langle \tilde{a}(t) \rangle = \langle \tilde{a}(s) \rangle e^{-\frac{\gamma}{2}(t-s)}$  ( $t \geq s$ ). According to the quantum regression theorem, we know that the time correlation functions satisfy the following equations ( $t \geq s$ ) [26,30]:

$$\frac{d}{dt} \langle \tilde{a}^\dagger(t) \tilde{a}(s) \rangle = -\frac{\gamma}{2} \langle \tilde{a}^\dagger(t) \tilde{a}(s) \rangle, \quad \frac{d}{dt} \langle \tilde{a}(t) \tilde{a}(s) \rangle = -\frac{\gamma}{2} \langle \tilde{a}(t) \tilde{a}(s) \rangle. \quad (\text{B15})$$

Here  $\langle \tilde{o}_1(t) \tilde{o}_2(s) \rangle$  are correlation functions in the rotating frame, defined by  $\langle \tilde{o}_1(t) \tilde{o}_2(s) \rangle = \text{tr}[\hat{o}_1 \mathcal{E}_{t-s} \hat{o}_2 \mathcal{E}_s \rho(0)]$  for  $t \geq s$  [26], where  $\hat{o}_{1,2}$  are operators in the Schrödinger picture and  $\mathcal{E}_t$  is the evolution operator solved from the above master equation in the interaction picture [Eq. (B12)], and  $\rho(t) = \mathcal{E}_{t-s} \rho(s)$ . Similarly,  $\langle \hat{o}_1(t) \hat{o}_2(s) \rangle$  are correlation functions without adopting the rotating frame, and we have

$$\begin{aligned} \langle \hat{a}^\dagger(t) \hat{a}(s) \rangle &= \langle \tilde{a}^\dagger(t) \tilde{a}(s) \rangle e^{i\Omega(t-s)} = \langle \tilde{a}^\dagger(s) \tilde{a}(s) \rangle e^{i\Omega(t-s)} e^{-\frac{\gamma}{2}(t-s)}, \\ \langle \hat{a}(t) \hat{a}(s) \rangle &= \langle \tilde{a}(t) \tilde{a}(s) \rangle e^{-i\Omega(t+s)} = \langle \tilde{a}^2(s) \rangle e^{-2i\Omega s} e^{-i\Omega(t-s)} e^{-\frac{\gamma}{2}(t-s)}. \end{aligned} \quad (\text{B16})$$

This can also be calculated using the Langevin equation  $\dot{\hat{a}} = -i\Omega \hat{a} - \frac{1}{2}\gamma \hat{a} - \hat{\xi}(t)$  (here we only consider one single bath), e.g.,

$$\begin{aligned} \langle \hat{a}^\dagger(t) \hat{a}(s) \rangle &= \left\langle \left[ \hat{a}^\dagger(s) e^{(i\Omega - \frac{\gamma}{2})(t-s)} - \int_s^t dt' e^{(i\Omega - \frac{\gamma}{2})(t-t')} \hat{\xi}^\dagger(t') \right] \hat{a}(s) \right\rangle \\ &= \langle \hat{a}^\dagger(s) \hat{a}(s) \rangle e^{(i\Omega - \frac{\gamma}{2})(t-s)} - \int_s^t dt' e^{(i\Omega - \frac{\gamma}{2})(t-t')} \left\langle \hat{\xi}^\dagger(t') \left[ \hat{a}(0) e^{(-i\Omega - \frac{\gamma}{2})s} - \int_0^s ds' e^{(-i\Omega - \frac{\gamma}{2})(s-s')} \hat{\xi}(s') \right] \right\rangle \\ &= \langle \hat{a}^\dagger(s) \hat{a}(s) \rangle e^{(i\Omega - \frac{\gamma}{2})(t-s)} + \int_s^t dt' \int_0^s ds' e^{(i\Omega - \frac{\gamma}{2})(t-t')} e^{(-i\Omega - \frac{\gamma}{2})(s-s')} \langle \hat{\xi}^\dagger(t') \hat{\xi}(s') \rangle. \end{aligned}$$

Under the Markovian approximation, we have  $\langle \hat{\xi}^\dagger(t') \hat{\xi}(s') \rangle \sim \delta(t' - s')$  [27,30]. In addition, notice that in the above double integral we have  $0 \leq s' \leq s \leq t' \leq t$ , thus the above integral gives zero.

### 4. Bath entropy change

Now we show the calculation for the second relation of Eq. (16) in the main text. Using the Heisenberg equation, we obtain

$$\frac{d}{dt} \sum_k \mathfrak{h}_k \langle \tilde{b}_{\alpha k}^2(t) \rangle + \text{H.c.} = \sum_k -i 2g_{\alpha k}^* \mathfrak{h}_k \left[ \langle \hat{a}(t) \hat{b}_{\alpha k}(0) \rangle e^{i\omega_{\alpha k} t} - i g_{\alpha k}^* \int_0^t ds e^{i\omega_{\alpha k}(t+s)} \langle \hat{a}(t) \hat{a}(s) \rangle \right] + \text{H.c.} \quad (\text{B17})$$

The first term in the brackets is further calculated by substituting  $\hat{a}(t)$  [Eq. (15)], and it gives

$$\begin{aligned} & - \sum_k 2|g_{\alpha k}|^2 \mathfrak{h}_k \int_0^t ds e^{-[i(\Omega - \omega_{\alpha k}) + \frac{\Gamma}{2}](t-s)} \langle \hat{b}_{\alpha k}^2(0) \rangle + \text{H.c.} \\ & \simeq 2 \int_0^\infty \frac{d\omega}{2\pi} \int_0^\infty ds J_\alpha(\omega) \mathfrak{h}(\omega) e^{-i(\Omega - \omega)s - \frac{1}{2}\Gamma s} \tilde{\mathbf{u}}_\alpha(\omega) + \text{H.c.} \\ & = 2 \int_0^\infty \frac{d\omega}{2\pi} J_\alpha(\omega) \left[ \frac{\mathfrak{h}(\omega) \tilde{\mathbf{u}}_\alpha(\omega)}{\frac{\Gamma}{2} + i(\Omega - \omega)} + \text{H.c.} \right] \simeq \gamma_\alpha [\mathfrak{h}(\Omega) \tilde{\mathbf{u}}_\alpha + \text{H.c.}]. \end{aligned} \quad (\text{B18})$$

Here we applied the Markovian approximation and the weak-coupling limit  $\Gamma \ll \Omega$ . The second term of Eq. (B17) can be calculated with the help of the relation (quantum regression theorem)

$$\langle \hat{a}(t) \hat{a}(s) \rangle = \langle \tilde{a}^2(s) \rangle e^{-2i\Omega s} e^{-i\Omega(t-s) - \frac{\Gamma}{2}(t-s)} \quad \text{for } t \geq s, \quad (\text{B19})$$

and it leads to

$$\begin{aligned}
& - \sum_k 2(g_{\alpha k}^*)^2 \hbar_k e^{2i\omega_k t} \int_0^t ds e^{-i\omega_k(t-s)} \langle \hat{a}(t) \hat{a}(s) \rangle + \text{H.c.} \\
& \simeq -2 \int \frac{d\omega}{2\pi} K_\alpha(\omega) \hbar(\omega) e^{2i(\omega-\Omega)t} \langle \tilde{a}^2(t) \rangle \int_0^\infty ds e^{[i(\Omega-\omega)-\frac{\Gamma}{2}]s} + \text{H.c.} \\
& = -2 \langle \tilde{a}^2(t) \rangle \int_0^\infty \frac{d\omega}{2\pi} \left[ \frac{K_\alpha(\omega) \hbar(\omega) e^{2i(\omega-\Omega)t}}{\frac{\Gamma}{2} + i(\omega - \Omega)} + \text{H.c.} \right] \simeq -\gamma_\alpha [\hbar(\Omega) \langle \tilde{a}^2(t) \rangle + \text{H.c.}].
\end{aligned} \tag{B20}$$

Thus, summing up Eqs. (B18) and (B20), we finish our calculation,

$$\frac{d}{dt} \sum_k \hbar_k \langle \tilde{b}_{\alpha k}^2(t) \rangle + \text{H.c.} = -\hbar(\Omega) \gamma_\alpha [\langle \tilde{a}^2(t) \rangle - \tilde{u}_\alpha] + \text{H.c.} \tag{B21}$$

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